SYMMETRIC TENSOR RANK AND THE IDENTIFICATION OF A POINT USING LINEAR SPANS OF AN EMBEDDED VARIETY

EDOARDO BALLICO
Department of Mathematics
University of Trento
38123 Povo (TN), Italy
Email: ballico@science.unitn.it

Abstract. Let \( X \subseteq \mathbb{P}^n \) be an integral and non-degenerate variety. Fix \( P \in \mathbb{P}^n \). In this paper we discuss the minimal integer \( \sum_{i=1}^{k} \sharp(S_i) \) such that \( S_i \subseteq X \) and \( \{ P \} = \cap_{i \geq 1} \langle S_i \rangle \), where \( \langle \rangle \) denote the linear span (in positive characteristic sometimes this integer is \( +\infty \)). We use tools introduced for the study of the \( X \)-rank of \( P \). Our main results are when \( X \) is a Veronese embedding of \( \mathbb{P}^m \) (it is related to the symmetric tensor rank of \( P \)) or when \( X \) is a curve.

1. Introduction

Let \( X \subseteq \mathbb{P}^n \) be an integral and non-degenerate variety defined over an algebraically closed field \( K \). For any \( P \in \mathbb{P}^n \) the \( X \)-rank \( r_X(P) \) of \( P \) is the minimal cardinality of a finite set \( S \subseteq X \) such that \( P \in \langle S \rangle \), where \( \langle \rangle \) denote the linear span. Let \( ir_X(P) \) be the minimal integer \( s \) such that there are finite sets \( S_i \subseteq X \), \( i \geq 1 \), such that \( \sharp(S_i) \leq s \) for all \( i \) and \( \{ P \} = \cap_{i \geq 1} \langle S_i \rangle \). We prove that \( ir_X(P) \) is less than \( +\infty \) if \( \text{char}(K) = 0 \) (Proposition 3), but we show that in positive characteristic this is not true in a few cases (Proposition 3). We call \( ir_X(P) \) the identification rank of \( P \) with respect to \( X \) or the \( X \)-identification rank of \( P \). Let \( \alpha(X,P) \) be the minimal integer \( x \) such that there are finitely many finite sets \( S_i \subseteq X \), say \( S_1, \ldots, S_k \), such that \( \{ P \} = \cap_{i=1}^{k} \langle S_i \rangle \) and \( \sum_{i=1}^{k} \sharp(S_i) = x \) (we don’t fix the integer \( k \) and we don’t assume that the sets \( S_i \) are disjoint, although the last condition is always satisfied if \( k = 2 \)). The integer \( \alpha(X,P) \) is the minimal number of points of \( X \) needed to identify \( P \) among all the points of \( \mathbb{P}^n \) using only the operations of linear algebra: first taking several linear spans of points of \( X \) and then taking the intersection of these linear subspaces. It is the analogous in projective geometry of the minimal number of photos needed to identify a point of \( \mathbb{R}^3 \). With a smaller number of points we may only identify a linear subspace, \( L \), containing \( P \), but we cannot distinguish \( P \) from the other points of \( \mathbb{P}^n \). One could allow both intersections and unions of
linear spaces $(S_i)$, $S_i \subset X$, but obviously in this way the minimal number $\sum_i \sharp(S_i)$ is at least the integer $\alpha(X,P)$ as we defined it. We say that $\alpha(X,P)$ is the \textit{identification number} of $P$ with respect to $X$. This concept has an obvious geometric meaning, but as in the case of the usual $X$-rank other related technical definitions may help to compute it. The integer $ir_X(P)$ is quite useful to get an upper bound for the integer $\alpha(X,P)$.

These two integers $ir_X(P)$ and $\alpha(X,P)$ are the key definitions introduced in this paper. We also add other related numerical invariants related to $ir_X(P)$ and $\alpha(X,P)$. We will see in the proofs that these invariants are quite useful to compute $ir_X(P)$ and $\alpha(X,P)$. First of all, several times it is important to look at zero-dimensional subschemes, not just finite sets, to take the linear span. This was a key ingredient for the study of binary forms ([14], [8], §3, [20], §4) and it is very useful also for multivariate polynomials ([8]). The \textit{cactus rank} $z_X(P)$ of $P$ with respect to $X$ is the minimal degree of a zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$ ([10], [9]). Let $iz_X(P)$ be the minimal integer $t$ such that there are zero-dimensional subschemes $Z_i \subset X$, $i \geq 1$, such that $\{P\} = \cap_i(Z_i)$. Obviously $iz_X(P) \leq ir_X(P)$ and $iz_X(P) = 1$ if and only if $P \in X$. Let $\gamma(X,P)$ be the minimal integer $x$ such that there are finitely many zero-dimensional schemes $Z_i \subset X$, say $Z_1, \ldots, Z_k$, such that $\{P\} = \cap_{i=1}^k(Z_i)$ and $\sum_{i=1}^k \deg(Z_i) = x$. Obviously

$$P \in X, \iff \alpha(X,P) \iff \gamma(X,P) = 1.$$ 

Most of our results are for curves and Veronese varieties (in the latter case the $X$-rank of $P$ is called the symmetric tensor rank of $X$) (see [2],[8],[15],[19],[20]). In the case of Veronese varieties we give a complete classification of the possible integers $ir_X(P)$, $iz_X(P)$ and $\alpha(X,P)$ when either $P$ has border rank 2 (Theorem 4) or $r_X(P) = 3$ (Theorem 5).

We prove the following results.

\textbf{Proposition 1.} Let $X \subset \mathbb{P}^{2k}$, $k \geq 1$, be an integral and non-degenerate curve. For a general $P \in \mathbb{P}^{2k}$ we have $r_X(P) = ir_X(P) = k + 1$ and $\alpha(X,P) = 2k + 2$.

\textbf{Theorem 1.} Assume $\text{char}(\mathbb{K}) = 0$. Let $X \subset \mathbb{P}^{2k+1}$ be an integral and non-degenerate curve. Fix a general $P \in \mathbb{P}^{2k+1}$.

(a) If $X$ is not a rational normal curve, then $r_X(P) = ir_X(P) = k + 1$ and $\alpha(X,P) = 2k + 2$.

(b) If $X$ is a rational normal curve, then $r_X(P) = z_X(P) = k + 1$, $ir_X(P) = iz_X(P) = k + 2$ and $\alpha(X,P) = \gamma(X,P) = 2k + 3$.

We also have a result on strange curves (Proposition 3), results on space curves (Theorems 2 and 3) and on rational normal curves (Propositions 5 and 6).

\section{2. Arbitrary characteristic}

For any integral variety $X \subset \mathbb{P}^n$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^n$ of the union of all linear spaces $\langle S \rangle$ with $S \subset X$ and $\sharp(S) = t$. Each $\sigma_t(X)$ is an integral variety, $\sigma_1(X) = X$ and $\dim(\sigma_t(X)) \leq \min\{n, t \cdot \dim(X) - 1\}$. For each $P \in \mathbb{P}^n$ the $X$-border rank $b_X(P)$ of $X$ is the minimal integer $t$ such that $P \in \sigma_t(X)$. Let $\tau(X) \subset \mathbb{P}^n$ denote the tangent developable of $X$, i.e. the closure in $\mathbb{P}^n$ of all tangent spaces $T_QX \subset \mathbb{P}^n$, $Q \in X_{\text{reg}}$. The algebraic set $\tau(X)$ is an integral variety,

$$\dim(\tau(X)) \leq \min\{n, 2 \cdot \dim(X)\}$$

and $\tau(X) \subset \sigma_2(X)$ (it is called the tangent developable of $X$).
Notation 1. For any linear subspace $V \subseteq \mathbb{P}^n$ let $\ell_V : \mathbb{P}^n \setminus V \to \mathbb{P}^{n-k-1}$, $k := \dim(V)$, denote the linear projection from $V$. If $V$ is a single point, $O$, we often write $\ell_O$ instead of $\ell_{\{O\}}$.

Notation 2. Let $\mathcal{Z}(X,P)$ (resp. $\mathcal{S}(X,P)$) denote the set of all zero-dimensional schemes $Z \subset X$ (resp. finite sets $S \subset X$) such that $\deg(Z) = z_X(P)$ (resp. $\sharp(S) = r_X(P)$) and $P \in \langle Z \rangle$ (resp. $P \in \langle S \rangle$).

As in [11], Lemma 2.1.5, and [8], Proposition 11, we use the following important invariant $\beta(X)$ of the embedded variety $X \subset \mathbb{P}^n$.

Notation 3. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Let $\beta(X)$ denote the maximal integer $t$ such that any zero-dimensional scheme $Z \subset X$ with $\deg(Z) \leq t$ is linearly independent, i.e. $\dim(\langle Z \rangle) = \deg(Z) - 1$.

Remark 1. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. Fix $P \in \mathbb{P}^n$. If $b_X(P) \leq \beta(X)$ and $X$ is either a smooth curve or a smooth surface, then $z_X(P) = b_X(P)$ ([11], Lemma 2.1.5, or [8], Proposition 11).

Take any integral and non-degenerate variety $X \subset \mathbb{P}^n$ and any finite set $S \subset X$ such that $\sharp(S) \leq \beta(X)$. By the definition of $\beta(X)$ the set $S$ is linearly independent. It seems better in Notation 3 to prescribe the linear independence of an arbitrary zero-dimensional scheme $Z \subset X$ with $\deg(Z) \leq \beta(X)$. Anyway, in many important cases (e.g. the Veronese varieties) the set-theoretic definition and the scheme-theoretic one chosen in Notation 3 give the same integer.

Remark 2. Obviously $\beta(X) \leq n + 1$ and equality holds if $X$ is a rational normal curve. We claim that equality holds if and only if $X$ is a rational normal curve. Indeed, if $X$ is a curve with degree $d \geq n + 1$, then a general hyperplane section of $X$ contains $d$ points spanning only a hyperplane. Now assume $\dim(X) \geq 2$. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Since $H \cap X$ is infinite, we may find $S \subset H \cap X$ with $\sharp(S) = n + 1$. Since $S$ is linearly dependent, $\beta(X) \leq n$ even in this case.

Remark 3. Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$ and $P \in \mathbb{P}^n$. Obviously $ir_X(P) = +\infty$ if and only if $ir_X(P) > n$. Since the intersection of $n-1$ hyperplanes of $\mathbb{P}^n$ contains at least a line, if $r_X(P) = ir_X(P) = n$, then $\alpha(X,P) = n^2$. We have $r_X(P) = n + 1$ if and only if $\dim(X) = 1$ and $X$ is a flat curve in the sense of [4]. Obviously if $r_X(P) = n + 1$, then $ir_X(P) = +\infty$. See [4], Proposition 1 and Example 1, for two classes of flat curves.

Let $X \subsetneq \mathbb{P}^n$ be an integral and non-degenerate variety and $P \in \mathbb{P}^n$. We say that $P$ is a strange point of $X$ if for a general $Q \in X_{reg}$ the Zariski tangent space $T_QX$ contains $P$ (we allow the case in which $X$ is a cone with vertex containing $P$). The strange set of $X$ is the set of all strange points of $X$ (this set is always a linear subspace, but usually it is empty). If this set is not empty, then either $\dim(\mathbb{K}) > 0$ or $X$ is a cone and the strange set of $X$ is the vertex of $X$ ([7],[22]). Lines and smooth conics in characteristic two are the only smooth strange curves ([17], Theorem IV.3.9). Now fix $P \in \mathbb{P}^n \setminus X$ and set $f_{P,X} := \ell_P|X$. Since $P \notin X$, $f_{P,X}$ is a finite morphism and we have $\deg(X) = \deg(f_{P,X}) \cdot \deg(f_{P,X}(X))$. The point $P$ is a strange point of $X$ if and only if $f_{P,X}$ is not separable. We recall that a non-degenerate curve $X \subset \mathbb{P}^n$, $n \geq 3$, is said to be very strange if a general hyperplane section of $X$ is not in linearly general position ([22]). A very strange curve is strange ([22], Lemma 1.1).
Proposition 2. Fix an integral and non-degenerate variety \( X \subseteq \mathbb{P}^n \). Set \( m := \dim(X) \) and fix \( P \in \mathbb{P}^n \). If \( P \) is not a strange point of \( X \), then \( \text{ir}_X(P) \leq n-m+1 \).

Proof. We will follow the proof of part (a) of [4], Theorem 1. If \( P \in X \), then \( \text{ir}_X(P) = 1 \). Hence we may assume \( P \notin X \). First assume \( m = 1 \). Let \( H \subset \mathbb{P}^n \) be a general hyperplane containing \( P \). Since \( P \) is not a strange point of \( X \), \( H \) is transversal to \( X \), i.e. \( H \cap \text{Sing}(X) = \emptyset \) and \( \sharp(H \cap X) = \deg(X) \). Since \( X \) is reduced and irreducible, we have \( h^1(\mathcal{I}_X) = 0 \). From the exact sequence

\[
0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X \cap H,H}(1) \to 0
\]

we get that the set \( H \cap X \) spans \( H \). Since \( P \in H \), we get the existence of \( S_H \subset X \cap H \) such that \( \sharp(S_H) \leq n \) and \( P \in \langle S_H \rangle \). Fix general hyperplanes \( H_i, \ i \leq i \leq n, \) containing \( P \) and such that \( \{P\} = H_1 \cap \cdots \cap H_n \). Take \( S_{H_i} \subset X \cap H_i \) as above. Since \( \{P\} = \bigcap_{i=1}^n \langle S_{H_i} \rangle \), we get \( \text{ir}_X(P) \leq n \). Now assume \( m \geq 2 \). We use induction on \( m \). Take a general hyperplane \( H \subset \mathbb{P}^n \) containing \( P \). Bertini’s theorem gives that \( X \cap H \) is geometrically integral ([18], part 4) of Th. I.6.3). Fix a general \( Q \in (X \cap H)_{\text{reg}} \). For general \( H \) we may take as \( Q \) a general point of \( X \). Hence \( P \notin T_QX \). Hence \( P \notin (T_QX) \cap H = T_Q(X \cap H) \). Thus \( P \) is not a strange point of \( X \cap H \). By the inductive assumption in \( H \cong \mathbb{P}^{n-1} \) we get \( \text{ir}_{X \cap H}(P) \leq n - m + 1 \). Since \( \text{ir}_X(P) \leq \text{ir}_{X \cap H}(P) \), we are done. \( \square \)

Proposition 3. Fix an integral and non-degenerate strange curve \( X \subset \mathbb{P}^n \). Fix \( P \in \mathbb{P}^n \setminus X \) and assume that \( P \) is the strange point of \( X \). Let \( s \) (resp. \( p \)) denote the separable (resp. inseparable) degree of \( f_{P,X} \). Set \( d := \deg(X) \) and \( c := \deg(f_{P,X}(X)) \). We have \( d = sp^c \).

(a) If \( s \geq 2 \), then \( \text{ir}_X(P) = 2 \).

(b) If \( s = 1, c \neq n-1 \) and \( X \) is not very strange, then \( \text{ir}_X(P) \leq n \).

(c) If \( s = 1 \) and \( c = n-1 \), then \( \text{ir}_X(P) = n+1 \) and \( \text{ir}_X(P) = +\infty \).

Proof. Since \( P \notin X \), \( f_{P,X} \) is a finite morphism. Hence \( \deg(X) = \deg(f_{P,X}) \cdot \deg(f_{P,X}(X)) \), i.e. \( d = sp^c \).

First assume \( s \geq 2 \). Fix general \( P_1, P_2 \in f_{P,X}(X) \). By assumptions there are \( O_{ij} \in f_{P_1P_2}(X), \ i = 1, 2, \ j = 1, 2, \) such that \( O_{11} \neq O_{12} \). Set \( S_i := \langle O_{11}, O_{12} \rangle \). Since \( P \in (S_i), \ i = 1, 2, \) and the two lines \( (S_i) \) are different, we get \( \text{ir}_X(P) = 2 \).

From now on we assume \( s = 1 \) and that \( X \) is not very strange. Let \( u : Y \to X \) denote the normalization map. Let \( \mathcal{H} \) be the set of all hyperplanes of \( \mathbb{P}^{n-1} \) transversal to \( f_{P,X}(X) \). We have \( \dim(\mathcal{H}) = n-1 \). Since \( f_{P,X}(X) \) is non-degenerate, we have \( \deg(f_{P,X}(X)) \geq n-1 \).

First assume \( c \neq n-1 \). Hence for every \( H \in \mathcal{H} \) we may find a set \( A_H \subset H \cap f_{P,X}(X) \) such that \( \sharp(A_H) = n \) and \( \langle A_H \rangle = H \). Notice that \( A_H \) is linearly dependent. Fix \( S_H \subset X \) such that \( \sharp(S_H) = n \) and \( f_{P,X}(S_H) = A_H \). If \( P \notin \langle S_H \rangle \), then \( S_H \) is linearly dependent. Since \( X \) is not very strange, we have \( X \cap \langle S \rangle = S \) (as sets) for a general set \( S \subset X \) such that \( \sharp(S) = n-1 \). Hence there is at most an \((n-2)\)-dimensional family of linearly dependent subsets of \( X \) with cardinality \( n \). Hence there is a non-empty open subset \( \mathcal{H}' \) of \( \mathcal{H} \) such that \( P \in \langle S_H \rangle \) for every \( H \in \mathcal{H}' \). Since \( \cap_{H \in \mathcal{H}'} H = \emptyset \), we get \( \{P\} = \cap_{H \in \mathcal{H}'} \langle S_H \rangle \). Hence \( \text{ir}_X(P) \leq n \).

Now assume \( c = n-1 \). Hence \( f_{P,X}(X) \) is a rational normal curve. In particular \( f_{P,X}(X) \) is smooth. Since \( f_{P,X} \circ u : Y \to f_{P,X}(X) \) is a purely inseparable morphism between smooth curves, it is injective. Hence \( f_{P,X} \) is injective. Since \( f_{P,X}(X) \) is a rational normal curve, for every \( S \subset X \) with \( \sharp(S) \leq n \), the set \( f_{P,X}(S) \) is a linearly
independent set with \( \sharp(S) \) elements. Hence \( P \notin \langle S \rangle \). Hence \( \mathfrak{r}_X(P) = n + 1 \). Hence \( \mathfrak{r}_X(P) > n \), i.e. \( \mathfrak{r}_X(P) = +\infty \).

All strange curves may be explicitly constructed (see \([7]\) for the case \( n = 2 \) and \([3]\) for the case \( n > 2 \)).

3. Curves

We use the following obvious observations (true in arbitrary characteristic) and whose linear algebra proof is left to the reader (parts (a) and (b) of Lemma 1 just say that two distinct lines have at most one common point and that if \( P \in \langle \{P_1, P_2\}\rangle \) and \( \mathfrak{r}_X(P) < 4 \), then there is \( S \subset X \) with \( \sharp(S) \leq 3 \), \( P \in \langle S \rangle \) and \( \langle \{P_1, P_2\}\rangle \not\subseteq \langle S \rangle \).

**Lemma 1.** Let \( X \subset \mathbb{P}^3 \) be an integral and non-degenerate curve. Fix \( P \in \mathbb{P}^3 \setminus X \).

(a) If \( \mathfrak{r}_X(P) = \mathfrak{i}_X(P) = 2 \), then \( \alpha(X, P) = 4 \).

(b) If \( \mathfrak{r}_X(P) = 2 \) and \( \mathfrak{i}_X(P) = 3 \), then \( \alpha(X, P) = 5 \).

(c) If \( \mathfrak{r}_X(P) = \mathfrak{i}_X(P) = 3 \), then \( \alpha(X, P) = 9 \).

**Remark 4.** Now assume that \( X \) is a singular curve, but take a zero-dimensional scheme \( Z \subset X_{\text{reg}} \) such that \( k := \deg(Z) \leq \beta(X)/2 \). Since \( Z \) is curvilinear, it has finitely many linear subschemes. Since \( Z \) is linearly independent, the set \( \Psi := \langle Z \rangle \backslash \langle Z' \rangle \) is a non-empty open subset of the \((k-1)\)-dimensional linear space \( \langle Z \rangle \). Fix any \( P \in \Psi \). Lemma 3 gives \( \sharp_X(P) = k \) and that \( Z \) is the only degree \( k \) subscheme of \( X \) whose linear span contains \( P \). Since \( Z \subset X_{\text{reg}} \), \( Z \) is smoothable. Hence \([8]\), Proposition 11, give \( b_X(P) = k \).

**Lemma 2.** Let \( X \subset \mathbb{P}^n \) be an integral and non-degenerate curve. Fix \( P \in \mathbb{P}^n \) such that \( \sharp_X(P) \leq \beta(X)/2 \). Then:

(i) There is a unique zero-dimensional scheme \( A \subset X \) such that \( P \in \langle A \rangle \) and \( \deg(A) \leq \sharp_X(P) \). We have \( \deg(A) = \sharp_X(P) \).

(ii) Fix any zero-dimensional scheme \( W \subset X \) such that \( \deg(W) \leq \beta(X) - \sharp_X(P) \) and \( P \in \langle W \rangle \). Then \( W \supseteq A \). We have \( \mathfrak{r}_X(P) \geq \mathfrak{i}_X(P) \geq \beta(X) - \sharp_X(P) + 1 \).

(iii) Assume that \( A \) is not reduced. Then \( \mathfrak{r}_X(P) \geq \beta(X) - \sharp_X(P) + 1 \). If \( \mathfrak{r}_X(P) = \beta(X) - \sharp_X(P) + 1 \), then \( S \cap A = \emptyset \) for all sets \( S \subset X \) such that \( \sharp(S) = \mathfrak{r}_X(P) \) and \( P \in \langle S \rangle \).

**Proof.** Assume the existence of zero-dimensional schemes \( A, W \) such that \( A \neq W \), \( P \in \langle A \rangle \cap \langle W \rangle \), \( P \notin \langle A' \rangle \) for all \( A' \subset A \) and \( \deg(A) + \deg(W) \leq \beta(X) \). Lemma 3 gives the existence of \( W' \supseteq W \) such that \( P \in \langle W' \rangle \). If \( W' \neq W \), then we continue taking \( W' \) instead of \( W \). We get parts (a) and (b).

The first assertion of part (iii) follows from part (ii), while the second one follows from Lemma 3.

**Proposition 4.** Let \( X \subset \mathbb{P}^3 \) be a rational normal curve. Then \( \mathfrak{r}_X(P) = 3 \) for all \( P \in \mathbb{P}^3 \setminus X \).

**Proof.** Lines and smooth conics in characteristic two are the only smooth strange curves ([17], Theorem IV.3.9). Fix \( P \in \mathbb{P}^3 \setminus X \). Since \( X \) is not strange, we have \( \mathfrak{r}_X(P) \leq 3 \) (Proposition 3) (even in positive characteristic). Since \( \sigma_2(X) = \mathbb{P}^3 \) ([1], Remark 1.6), Remark 3 gives \( \sharp_X(P) = 2 \). Since \( \beta(X) = 4 \), Lemma 3 gives \( \mathfrak{r}_X(P) \geq 3 \).
Let \( X \) be a smooth elliptic curve defined over \( \mathbb{K} \). We recall that the 2-rank of \( X \) is the number, \( \epsilon \), of pairwise non-isomorphic line bundles \( L \) on \( X \) such that \( L^{\otimes 2} \cong \mathcal{O}_X \) ([23], Chapter III). If \( \text{char}(\mathbb{K}) \neq 2 \), then \( \epsilon = 4 \), while \( \epsilon \in \{1, 2\} \) if \( \text{char}(\mathbb{K}) = 2 \) ([23], Corollary III.6.4).

**Theorem 2.** Let \( X \subset \mathbb{P}^3 \) be a smooth elliptic curve. Fix \( P \in \mathbb{P}^3 \setminus X \). Let \( \epsilon \) be the 2-rank of the elliptic curve \( X \). There are exactly \( \epsilon \) quadric cones \( W_i \), \( 1 \leq i \leq \epsilon \), containing \( X \). Call \( O_i \), \( 1 \leq i \leq \epsilon \), the vertex of \( W_i \).

(a) The points \( O_i \), \( 1 \leq i \leq \epsilon \), are the only points \( Q \in \mathbb{P}^3 \) such that \( Z(X, P) \) and \( S(X, Q) \) are infinite; we have \( \pi_X(O_i) = 2 \) for all \( i \); each point \( O_i \) is contained in \( TX \).

(b) If \( P \in (TX \cup \bigcup_{i=1}^{\epsilon} W_i) \), but \( P \neq O_i \) for any \( i \), then \( \pi_X(P) = 3 \).

(c) If \( P \notin (TX \cup \bigcup_{i=1}^{\epsilon} W_i) \), then \( \pi_X(P) = 2 \).

**Proof.** Call \( R_i \), \( 1 \leq i \leq \epsilon \), the pairwise non-isomorphic line bundles on \( X \) such that \( R_i^{\otimes 2} \cong \mathcal{O}_X \). Since \( \text{deg}(X) \) is even and \( \mathbb{K} \) is algebraically closed, there is a line bundle \( L \) on \( X \) such that \( L^{\otimes 2} \cong \mathcal{O}_X(1) \). Set \( L_i := R_i \otimes L \). It is easy to check that the line bundles \( L_i \), \( 1 \leq i \leq \epsilon \), are pairwise non-isomorphic and that, up to isomorphisms, they are the only line bundles \( A \) on \( X \) such that \( A^{\otimes 2} \cong \mathcal{O}_X(1) \).

Since \( X \) is not strange, Proposition 3 gives \( \pi_X(P) \leq 3 \). Since \( P \notin X \), Remark 3 and [1], Remark 1.6, give \( \pi_X(P) = 2 \). Obviously, if \( \pi(X, P) = 1 \), then \( \pi_X(P) > 2 \). Since \( \ell_P(X) \) spans \( \mathbb{P}^2 \), we have \( \text{deg}(\ell_P(X)) \geq 2 \). Hence either \( \text{deg}(\ell_P(X)) = 4 \) and \( \ell_P|X \) is birational onto its image or \( \text{deg}(\ell_P|X) = 2 \).

First assume \( \text{deg}(\ell_P|X) = 2 \). In this case we get that \( Z(X, P) \) is infinite. Since \( \ell_P(X) \cong \mathbb{P}^1 \), the morphism \( \ell_P|X \) is not purely inseparable. Hence a general fiber of it is formed by two distinct points of \( X \) spanning a line through \( P \). Hence \( \pi_X(P) = 3 \). We get \( \mathcal{O}_X(1) \cong \ell_P(\mathcal{O}_{\ell_P(X)}(1)) \). Since \( \mathcal{O}_{\ell_P(X)}(1) \cong R^{\otimes 2} \) with \( R \) a degree 1 line bundle on \( \ell_P(X) \), \( \ell_P(R) \) is one of the line bundle \( L_i \), \( 1 \leq i \leq \epsilon \). Since \( X \neq \mathbb{P}^1 \), \( \ell_P|X \) has at least one ramification point. Hence \( O_i \in TX \) for all \( i \). The construction may be inverted in the following sense. Fix one of the line bundles \( L_i \), \( 1 \leq i \leq \epsilon \). Since \( X \) is an elliptic curve, we have \( h^0(X, L_i) = 2 \) and the linear map \( j : S^2(H^0(X, L_i)) \to H^0(X, \mathcal{O}_X(1)) \) is injective with as image a hyperplane of the 4-dimensional linear space \( H^0(X, \mathcal{O}_X(1)) \), i.e. (by the linear normality of \( X \)) a point, \( O_i \), of \( \mathbb{P}^3 = \mathbb{P}(H^0(X, \mathcal{O}_X(1))^\vee) \). The definition of \( j \) gives that \( \ell_{O_i}|X \) has degree 2.

Now assume \( \text{deg}(\ell_P(X)) = 4 \). The genus formula for plane curves gives that \( \ell_P(X) \) has 1 or 2 singular points and that if it has two singular points, then they are either ordinary nodes or ordinary cusps. If \( \ell_P(X) \) has either a unique singular point or at least one cusp, then \( \pi_X(P) > 2 \) and hence \( \pi_X(P) = 3 \). In particular this is the case if \( P \in TX \). Hence if \( P \in TX \) and \( P \neq O_i \), then \( \pi_X(P) = 3 \). Now assume \( P \notin TX \). In this case \( \pi_X(P) = 2 \) if and only if \( \ell_P(X) \) has two singular points. If the plane curve \( \ell_P(X) \) has a unique singular point, then it is an ordinary tacnode. Let \( T \subset \mathbb{P}^3 \) be a line secant to \( X \), but not tangent to \( X \). Since \( X \) is the complete intersection of two quadric surfaces, there is a unique quadric surface, \( W \), containing \( X \cup \{P\} \). Call \( T \) a line in \( W \) containing \( P \). \( X \cup T \) is contained in a unique quadric surface, \( W \). If \( W \) is singular, i.e. if \( W = W_i \) for some \( i \), then there is a unique line through \( P \) and secant to \( X \). If \( W \) is smooth, i.e. if \( P \notin W_i \) for any \( i \), then there are two such lines, both of them containing two distinct points of \( X \), because we assumed \( P \notin TX \). Hence \( \pi_X(P) = 2 \) in this case.
Theorem 3. Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Assume that $X$ is not strange and that $X$ has only planar singularities. There is a non-empty open subset $\Omega$ of $\mathbb{P}^3 \setminus X$ such that $ir_X(P) = 2$ for all $P \in \Omega$ if and only if $X$ is not a rational normal curve.

Proof. Set $d := \deg(X)$ and $q := p_a(X)$. Since Proposition 4 gives that “only if” part, it is sufficient to prove the “if” part. Assume $d \geq 4$. It is easy to check the existence of a non-empty open subset $W$ of $\mathbb{P}^3 \setminus X$ such that $\ell_P|X$ is birational onto its image for all $P \in W$. By assumption for each $O \in \text{Sing}(X)$ the Zariski tangent plane $T_OX$ at $O$ is a plane. Since $\text{Sing}(X)$ is finite, we get finitely many planes $T_OX$, $O \in \text{Sing}(X)$, and we call $W'$ the intersection of $W$ with the complement of the union of these planes. Let $G$ be the intersection of $W'$ with the complement of the tangent developable $\tau(X)$ of $X$. For each $P \in G$ the morphism $\ell_P|X$ is unramified and birational onto its image. Hence the singularities of the degree $d$ plane curve $\ell_P(X)$ comes only from the non-injectivity of $\ell_P|X$ and the singularities of $X$. To prove Theorem 3 it is sufficient to prove that the set of all $P \in G$ such that $\ell_P|X$ has at least two fibers with cardinality $\geq 2$ contains a non-empty open subset. For any $O \in \text{Sing}(X)$ let $C_O(X)$ the cone with vertex $O$ and the plane curve $\ell_O(X \setminus \{O\})$ as its base. Set $G' := G \setminus (\cup_{Q \in \text{Sing}(X)} C_O(X))$. The set $G'$ is a non-empty open subset of $G$ and for every $P \in G'$ no point of $X \setminus \text{Sing}(X)$ is mapped onto a point of $\ell_P(\text{Sing}(X))$. Hence for each $P \in G'$ the plane curve $\ell_P(X)$ has $\#(\text{Sing}(X))$ singular points isomorphic to the corresponding singular points of $X$, plus some other singular points and the integer $p_a(\ell_P(X)) - q = (d-1)(d-2)/2 - q$ is the sum of the contributions of the other singular points. Since $X$ is not strange, it is not very strange, i.e. a general secant line of $X$ contains only two points of $X$ ([22], Lemma 1.1). This is equivalent to the existence of a non-empty open subset $G''$ of $G'$ such that for all $P \in G''$ each singular point of $\ell_P(X) \setminus \ell_P(\text{Sing}(X))$ has only two branches.

Claim: There is a non-empty open subset $G_1$ of $G''$ such that for every $P \in G_1$, $\ell_P(X) \setminus \ell_P(\text{Sing}(X))$ has only ordinary double points as singularities.

Proof of the Claim: Fix $P \in G''$. Fix $O \in \ell_P(X) \setminus \ell_P(\text{Sing}(X))$. By the definition of $G''$ there are exactly two points $Q_1, Q_2 \in X$ such that $\ell_P(Q_1) = \ell_P(Q_2) = O$, $X$ is smooth at $Q_1$ and $Q_2$, and $\ell_P|X$ is unramified at each $Q_i$. Hence $\ell_P(X) \setminus \ell_P(\text{Sing}(X))$ has only ordinary double points as singularities if and only if $\ell_P(T_{Q_1}X) \neq \ell_P(T_{Q_2}X)$, i.e. if and only if the planes $\langle \{P\} \cup T_{Q_i}X \rangle$, $i = 1, 2$, are distinct. This is certainly true if $T_{Q_1}X \cap T_{Q_2}X = \emptyset$. Let $V$ denote the set of all $(Q_1, Q_2) \in (X \setminus \text{Sing}(X)) \times (X \setminus \text{Sing}(X))$ such that $Q_1 \neq Q_2$. Let $U$ be the set of all $(Q_1, Q_2) \in V$ such that $T_{Q_1}X \cap T_{Q_2}X \neq \emptyset$. Since $X$ is not strange, $U$ is a union of finitely many subvarieties of dimension $\leq 1$; it is here that we use the full force of our assumption “ $X$ not strange ”, not only the far weaker condition “ $X$ not very strange ”. Let $\Delta$ be the closure in $\mathbb{P}^3$ of the union of the lines $\langle \{Q_1, Q_2\} \rangle$ with $(Q_1, Q_2) \in U$. We have $\dim(\Delta) \leq 2$. Set $G_1 := G'' \cap (\mathbb{P}^3 \setminus \Delta)$. By construction this set $G_1$ satisfies the Claim.

Now we prove that we may take $\Omega := G_1$. Fix $P \in G_1$ and call $x$ the number of the singular points of $\ell_P(X) \setminus \ell_P(\text{Sing}(X))$. By the claim it is sufficient to prove the inequality $x \geq 2$. Since $\ell_P(X)$ is a plane curve of degree $d$, it has arithmetic genus $(d-1)(d-2)/2$. Since each point of $\ell_P(X) \setminus \ell_P(\text{Sing}(X))$ is an ordinary
node, \( \ell_P \rvert X \) is unramified at each point of \( \text{Sing}(X) \) and \( \ell_P^{-1}(\ell_P(X) \setminus \ell_P(\text{Sing}(X))) \), we have \( x = p_a(\ell_P(X)) - p_a(X) = (d-1)(d-2)/2 - q \). Hence it is sufficient to prove that \( q \leq (d-1)(d-2)/2 - 2 \). This is true by the assumption \( d \geq 4 \) and Castelnuovo’s inequality for the arithmetic genus of space curves (use [22], Lemma 1.1, that \( X \) is not strange and that the upper bound needs only that a general plane section of \( X \) is in linearly general position). □

Proof of Proposition 1: Let \( \Delta \) denote the set of all linearly independent subsets of \( X \) with cardinality \( k+1 \). Since \( \sigma_{k+1}(X) = \mathbb{P}^{2k} \) and \( \dim(\sigma_k(X)) = 2k - 1 \) ([1], Remark 1.6), we have \( r_X(P) = k+1 \). A dimensional count gives that \( S(X, P) \) has a one-dimensional irreducible component, \( \Gamma \). Fix \( A, B \in \Gamma \). It is sufficient to prove that \( \langle P \rangle = \langle A \rangle \cap \langle B \rangle \). Since any two \( k \)-dimensional linear subspaces meet, the set \( A \) may be seen as a general element of \( \Delta \) and, after fixing \( A, P \) may be seen as a general element of \( \langle A \rangle \). Hence it is sufficient to prove that \( \langle A \rangle \cap \langle B \rangle \) is a single point for a general \( (A, B) \in \Delta \times \Delta \), i.e. to check that \( A \cup B \) spans \( \mathbb{P}^{2k} \). For fixed \( A \), we have \( \langle A \cup B \rangle = \mathbb{P}^{2k} \) for a general \( B \subset X \), because \( X \) spans \( \mathbb{P}^{2k} \).

Proof of Theorem 1: Since \( \sigma_{k+1}(X) = \mathbb{P}^{2k+1} \) and \( P \) is general, we have \( r_X(P) \leq k + 1 \) ([1], Remark 1.6). Since \( \dim(\sigma_k(X)) = 2k - 1 \) ([1], Remark 1.6) and \( P \) is general, we have \( r_X(P) \geq k + 1 \). Hence \( r_X(P) = k + 1 \). \( X \) is not a rational normal curve if and only if there are \( S_1, S_2 \subset X \) such that \( S_1 \neq S_2 \), \( \sharp(S_1) = \sharp(S_2) = k + 1 \) and \( P \in \langle S_1 \rangle \cap \langle S_2 \rangle \) ([13], Theorem 3.1). Let \( \Omega \) be the set of all \( Q \in \mathbb{P}^{2k+1} \setminus \sigma_k(X) \) such that there are only finitely many sets \( S \subset X \) with \( \sharp(S) = k + 1 \) and \( Q \in \langle S \rangle \). \( \Omega \) is a non-empty open subset of \( \mathbb{P}^{2k+1} \). Since \( P \) is general, we may assume \( P \in \Omega \).

(i) In this step we assume that \( X \) is not a rational normal curve. Let \( \Gamma \) denote the set of all finite sets \( S \subset X \) such that \( \sharp(S) = k + 1 \) and \( \dim(\langle S \rangle) = k \). We proved the existence of \( S_i \in \Gamma, i = 1, 2 \), such that \( P \in \langle S_1 \rangle \cap \langle S_2 \rangle \). To prove part (a) it is sufficient to prove that \( \langle P \rangle = \langle S_1 \rangle \cap \langle S_2 \rangle \) for a general \( P \). Assume that this is not true, i.e. assume that \( \langle S_1 \rangle \cap \langle S_2 \rangle \) is a linear space of dimension \( \rho > 0 \). Notice that \( S(X, P) = \{ S \in \Gamma : P \in \langle S \rangle \} \). Set \( \Gamma(S_1) := \{ S \in \Gamma : S \cap S_1 = \emptyset, (S \cap S_1) \cap \Omega \neq \emptyset \} \). Since \( \dim(S_1) = k \) and \( P \in \Omega \cap \langle S_1 \rangle \), then \( \Gamma(S_1) \neq \emptyset \) and \( \Gamma(S_1) \) has pure dimension \( k \). Since \( P \) is general in \( \mathbb{P}^{2k+1} \), we may assume that \( S_1 \) is general in \( \Gamma \) and that \( S_2 \) is general in one of the irreducible components of \( \Gamma(S_1) \). We get that for a general \( P' \in \Omega \cap \langle S_1 \rangle \) there is a \( \rho \)-dimensional family of sets \( S \) with \( P' \in \langle S \rangle \), absurd.

(ii) In this step we assume that \( X \) is a rational normal curve. We know that \( r_X(P) = k + 1 \). We proved that \( \alpha(X, P) \geq k + 2 \) and hence that \( \alpha(X, P) \geq 2k + 3 \). For a sufficiently general \( P \in \mathbb{P}^{2k+1} \) we call \( S_P \) the only subset of \( X \) with cardinality \( k+1 \) and whose linear span contains \( P \). Since \( \beta(X) = 2k + 2 \) and \( P \notin \sigma_k(X) \), Remark 3 gives \( z_X(P) = k + 1 \) and that \( S_P \) is the only degree \( k+1 \) zero-dimensional subscheme of \( X \) whose linear span contains \( P \). Hence \( iz_X(P) \geq k + 2 \) and \( \gamma(X, P) \geq 2k + 3 \).

Fix a general \( Q \in X \) and let \( \phi : X \rightarrow \mathbb{P}^{2k} \) denote the morphism induced from \( \ell_Q \rvert (X \setminus \{Q\}) \). The morphism \( \phi \) is an embedding of \( X \cong \mathbb{P}^1 \) as a rational normal curve of \( \mathbb{P}^{2k} \). Fix a general \( P' \in \mathbb{P}^{2k} \). Proposition 1 gives the existence of \( A_1, A_2 \subset \phi(X) \) such that \( \sharp(A_1) = \sharp(A_2) = k + 1 \) and \( \langle A_1 \rangle \cap \langle A_2 \rangle = \langle P' \rangle \). For a fixed point \( \phi(Q) \), but for general \( P' \) we may also assume \( \phi(Q) \notin \langle A_1 \cup A_2 \rangle \). Hence there is a unique set \( B_i \subset X \setminus \{Q\} \) such that \( \phi(B_i) = A_i \). Set \( E_i := \{Q\} \cup B_i \). Fix \( P'' \in \mathbb{P}^{2k+1} \) such that \( \ell_Q(P'') = P' \). For fixed \( Q \), but general \( P' \) we may consider
$P''$ as a general point of $\mathbb{P}^{2k+1}$. We have $\langle\{Q, P''\}\rangle = \langle E_1 \rangle \cap \langle E_2 \rangle$. Varying $Q$ in $X$ we get $ir_X(P) \leq k+2$ and hence $ir_X(P) = k+2$. Let $\Theta$ be the set of all finite subsets $A \subset X$ such that $z(A) = k+2$ and $P \in \langle A \rangle$. Assume for the moment the existence of $A \in \Theta$ such that $A \cap S_p = \emptyset$, i.e. such that $z(A \cup S_p) = 2k+3$. Since $\beta(X) = 2k+2$ and $z(A \cup S_p) = 2k+3$, we get $\langle S_p \cup A \rangle = \mathbb{P}^{2k+1}$, i.e. $\dim(\langle A \rangle \cap \langle S_p \rangle) = 0$ (Grassmann’s formula). Since $P \in \langle A \rangle \cap \langle S_p \rangle$, we get $\{P\} = \langle A \rangle \cap \langle S_p \rangle$, i.e. $\alpha(X, P) \leq 2k+3$. Hence $\alpha(X, P) = \gamma(X, P) = 2k+3$. Now assume $A \cap S_p \neq \emptyset$ for all $A \in \Theta$. Since $P$ is general and $\sigma_{k+2}(X) = \mathbb{P}^{2k+1}$, Terracini’s lemma (or a dimensional count) gives $\dim(\Theta) = 2$. For any $Q \in S_p$ set $\Theta_Q := \{A \in \Theta : Q \in A\}$.

The proof of the inequality $ir_X(P) \leq 2k+3$ also shows $\dim(\Theta_Q) = 1$. Since $S_p$ is finite, we get $\dim(\Theta) = 1$, a contradiction. 

\section{Veronese varieties}

For all integers $m \geq 1$ and $d \geq 1$ let $\nu_d : \mathbb{P}^m \to \mathbb{P}^n$, $n := (m+d) - 1$ denote the order $d$ embedding of $\mathbb{P}^m$ induced by the vector space of all degree $d$ homogeneous polynomials in $d + 1$ variables. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$.

We often use the following elementary lemma ([5], Lemma 1).

Lemma 3. Fix any $P \in \mathbb{P}^n$ and two zero-dimensional subschemes $A, B$ of $\mathbb{P}^n$ such that $A \neq B$, $P \in \langle A \rangle$, $P \not\in \langle B \rangle$, $P \not\in \langle A' \rangle$ for any $A' \subset A$ and $P \not\in \langle B' \rangle$ for any $B' \subset B$. Then $h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) > 0$.

We first need the case $m = 1$ of Theorem 4, i.e. we need to study the case in which $X$ is a rational normal curve (Propositions 5,6 and 7).

Proposition 5. Let $X \subset \mathbb{P}^d$, $d \geq 3$, be a rational normal curve. Fix a set $A \subset X$ with $z(A) = 2$ and any $P \in \langle A \rangle \setminus A$. Then $r_X(P) = z_X(P) = 2$, $ir_X(P) = iz_X(P) = d$ and $\alpha(X, P) = \gamma(X, P) = d + 2$. Moreover, there is a set $B \subset X$ such that $z(B) = d$ and $\{P\} = \langle A \rangle \cap \langle B \rangle$.

Proof. Since $\beta(X) = d + 1 \geq 3$, we have $A = \langle A \rangle \subset X$. Hence $P \not\in X$. Hence $ir_X(P) = 2 = iz_X(P)$. Fix a zero-dimensional scheme $W \subset X$ such that $P \in \langle W \rangle$, $P \not\in \langle W' \rangle$ for any $W' \subset W$ and $W \neq A$. Since $\beta(X) = d + 1$, Lemma 3 gives $\text{deg}(W) \geq d$. Hence $ir_X(P) \geq iz_X(P) \geq d$ and $\alpha(X, P) \geq \gamma(X, P) \geq d + 2$. Hence to conclude the proof it is sufficient to find a set $B \subset X$ such that $z(B) = d$ and $\{P\} = \langle A \rangle \cap \langle B \rangle$. Set $Y := \ell_P(X)$. Since $P \in \langle A \rangle$ and $P \not\in X$, the curve $Y$ is a linearly normal curve with degree $d$, arithmetic genus $1$ and a unique singular point, which is an ordinary node. Fix a general hyperplane $H \subset \mathbb{P}^{d-1}$ and set $E := Y \cap X$. Since $H$ is general, it does not contain the singular point of $Y$ and it is transversal to $Y$. Hence $E$ is a set of $d$ points and there is $B \subset X$ such that $z(B) = d$ and $\ell_P(B) = E$. Since $z(B) \leq \beta(X)$, $B$ is linearly independent. Since $E$ is linearly dependent, we have $P \in \langle B \rangle$. Since $z(A \cup B) = d + 2 = \beta(X) + 1$, we have $\langle A \cup B \rangle = \mathbb{P}^d$. Hence Grassmann’s formula gives $\{P\} = \langle A \rangle \cap \langle B \rangle$.

Proposition 6. Let $X \subset \mathbb{P}^d$, $d \geq 3$, be a rational normal curve. Fix $P \in \tau(X) \setminus X$, i.e. fix $P \in \sigma_2(X)$ such that $r_X(P) > 2$. Then $z_X(P) = 2$, $iz_X(P) = d$, $\alpha(X, P) = d + 2$, $r_X(P) = d$, $ir_X(P) = d$ and $\alpha(X, P) = d^2$. Moreover, there are a zero-dimensional $A \subset X$ and a finite set $B \subset X$ such that $\text{deg}(A) = 2$, $z(B) = d$ and $\{P\} = \langle A \rangle \cap \langle B \rangle$.

Proof. First of all we explain the “ i.e.” part. Since $\beta(X) \geq 2$, Remark 3 gives that for each $Q \in \sigma_2(X) \setminus X$ there is a degree $2$ zero-dimensional scheme $A_Q \subset X$
such that $Q \in \langle A_Q \rangle$. Since $\beta(X) \geq 4$, we also get the uniqueness of $A_Q$. Hence $P \in \tau(X) \iff A_P$ is not reduced $\iff r_X(P) > 2$. Set $A := A_P$. Lemma 3 gives $r_X(P) \geq d$ and $i \gamma_X(P) \geq d$. We repeat the proof of Proposition 5 (now $Y$ is a degree $d$ linearly normal curve with a cusp). We get the existence of a set $B \subset X$ such that $\sharp(B) = d$ and $\{P\} = \langle A \rangle \cap \langle B \rangle$. Hence $i \gamma_X(P) = d$, $\gamma(X, P) = d$. Since $d \geq 3$, $X$ is not strange. Hence $i \gamma_X(P) \leq d$ (Proposition 3). Since $r_X(P) \geq d$, we get $r_X(P) = i \gamma_X(P) = d$. Since $r_X(P) = d$, $P$ is contained in no linear space of dimension $\leq d-2$ spanned by a finite subset of $X$. Hence $\alpha(X, P) = d^2$ (Remark 3).

Proposition 7. Let $X \subset \mathbb{P}^d$, $d \geq 5$, be a rational normal curve. Fix a set $A \subset X$ such that $\sharp(A) = 3$ and any $P \in \langle A \rangle$ such that $P \notin \langle A' \rangle$ for any $A' \subset A$. Then $r_X(P) = i \gamma_X(P) = \sharp(A) = 3$.

Proof. Since $\beta(X) \geq 5$, Lemma 3 gives $z_X(P) = 3$, $i \gamma_X(P) \geq \beta(X)+1-\sharp(A) = d-1$ and hence $r_X(P) = 3, i \gamma_X(P) \geq d-1, \alpha(X, P) \geq \gamma(X, P) \geq d+2$.

Set $Y := \ell_P(X)$. Since $\beta(X) = d+1 \geq 5$ and $P \notin \langle A \rangle$ for any $A' \subset A$, $\ell_P|X$ is an embedding. Hence $Y$ is a smooth rational curve of degree $d$ spanning $\mathbb{P}^{d-1}$. Fix any $E \subset X \setminus A$ with $\sharp(E) = d-4$ and set $F := \ell_P(E)$. Since $\sharp(A \cup E) \leq \beta(X)$, $F$ is a set of $d-4$ points of $Y$ spanning a $(d-5)$-dimensional linear subspace disjoint from the line $\langle \ell_P(A) \rangle$.

Claim: For general $E$ we have $\langle E \rangle \cap Y = F$ (as schemes) and $\ell(\langle E \rangle)|Y \setminus F)$ extends to an embedding $\phi : Y \to \mathbb{P}^3$ with $\phi(Y) \subset \mathbb{P}^3$ a smooth and rational curve of degree 4 with $\phi(\ell_P(A))$ the union of 3 distinct and collinear points.

Proof of the Claim: The map $\phi$ is induced by the linear projection of $X$ from the linear subspace $\langle \{P\} \cup E \rangle$. Since $E \cap A = \emptyset$ and $\sharp(E \cup A) \leq \beta(X)$, we have $\langle E \rangle \cap \langle A \rangle = \emptyset$. Hence $\phi(A)$ is the union of 3 distinct collinear points. For degree reasons we get $\langle E \rangle \cap Y = F$ (as schemes), i.e. $\deg(\phi) \cdot \deg(\phi(Y)) = \deg(Y) - d+4 = 4$. Since $\phi(Y)$ spans $\mathbb{P}^3$, we get $\deg(\phi) = 1$. Since $\phi(Y)$ has a 3-secant line, the curve $Y$ is not the complete intersection of two quadric surfaces. Hence $\phi(Y)$ is smooth and rational.

Since $h^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2)) = 10 = h^0(\mathbb{P}^1, O_{\mathbb{P}^1}(8)) + 1$, the Claim implies the existence of a quadric surface $T$ containing $\phi(Y)$. Since $\phi(Y)$ has genus $\neq 1$, $T$ is not a cone ([17], V.2.9). Hence $\phi(Y)$ is a curve of type $(1,3)$ on the smooth quadric surface $T$. The set $\phi(\ell_P(A))$ is contained in a line of type $(1,0)$. Let $G$ be the intersection of $\phi(Y)$ with a general line of type $(1,0)$ of $T$. Since any two different lines of $T$ are disjoint, we have $\phi(A) \cap G = \emptyset$. Since $\phi(\ell_P(A))$ is reduced, in arbitrary characteristic we get that $G$ is reduced. Since the set $\phi(F)$ is finite, for a general line of type $(1,0)$ on $T$ we have $G \cap \phi(F) = \emptyset$. Hence there is $G' \subset Y \setminus F$ such that $\phi(G') = G$. Let $B \subset X$ be the only set such that $\ell_P(B) = F \cup G'$. Since $\sharp(B) \leq \beta(X)$, we have $\dim(\langle B \rangle) = d-2$. Since $G$ is linearly dependent, $F \cup G'$ is linearly dependent. Hence $P \in \langle B \rangle$. Since $A \cap B = \emptyset$ and $\beta(X) = d+1 \leq \sharp(A \cup B)$, we have $\langle A \cup B \rangle = \mathbb{P}^d$. Hence Grassmann’s formula gives that $\langle A \rangle \cap \langle B \rangle$ is a single point. Hence $\{P\} = \langle A \rangle \cap \langle B \rangle$. Hence $i \gamma_X(P) \leq d-1$ and $\alpha(X, P) \leq d + 2$. Since we proved the opposite inequalities, we are done.

Theorem 4. Fix integers $m \geq 1$ and $d \geq 3$. Set $n := n_{m,d} := \binom{m+d}{m} - 1$ and $X := X_{m,d}$. Fix $P \in \sigma_2(X_{m,d}) \setminus X$. 

©2014 albanian-j-math.com
(a) Assume $P \notin \tau(X)$, i.e., assume $r_X(P) = 2$. Then $ir_X(P) = d$, $z_X(P) = 2$, $iz_X(P) = d$ and $\alpha(X, P) = \gamma(X, P) = d + 2$.

(b) Assume $P \in \tau(X) \setminus X$. Then $z_X(P) = 2$, $iz_X(P) = ir_X(P) = d$, $\gamma(X, P) = d + 2$. If $m = 1$, then $\alpha(X, P) = d^2$. If $m \geq 2$, then $\alpha(X, P) = 3d$.

Proof. Since $d \geq 3$, we have $\sigma_2(X) \neq \tau(X)$, $\sigma_2(X) \setminus \tau(X) = \{ P \in \sigma_2(X) : r_X(P) = 2 \}$ and $r_X(P) = d$ for each $P \in \tau(X) \setminus X$ ([8], Theorem 32). Since the case $m = 1$ is true (Propositions 5 and 6), we assume $m \geq 2$. Since $\beta(X) = d + 1$ (e.g. by [8], Lemma 34), Remark 3 and Lemma 3 imply the existence of a unique zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = 2$ and $P \in \langle Z \rangle$. We have $r_X(P) = 2$ if and only if $Z$ is reduced. Let $A \subset \mathbb{P}^m$ be the degree 2 zero-dimensional scheme such that $\nu_d(A) = Z$. Let $L \subset \mathbb{P}^m$ be the line spanned by $A$. Set $R := \nu_d(L)$. Since $Z \subset R$, we have $r_X(P) \leq r_R(P)$, $z_X(P) \leq z_R(P)$, $ir_X(P) \leq ir_R(P)$, $iz_X(P) \leq iz_R(P)$, $\alpha(X, P) \leq \alpha(R, P) = d$ and $\gamma(X, P) \leq \gamma(R, P)$. Propositions 5 and 6 give $ir_R(P) = iz_R(P) = d$ and $\gamma(R, P) = d + 2$. Let $W \subset \mathbb{P}^m$ be a zero-dimensional scheme such that $P \in \langle \nu_d(W) \rangle$, $P \notin \langle \nu_d(W') \rangle$ for any $W' \subset W$ and $W \neq A$. Since $\beta(X) \geq d + 1$, Lemma 3 gives $\deg(W) \geq d$. Hence $iz_X(P) \geq d$ and $\gamma(X, P) \geq d + 2$. Hence $ir_X(P) = iz_X(P) = d + 2$ and $\gamma(X, P) = d + 2$. In case (a) we have $\alpha(X, P) = d + 2$, because $\alpha(R, P) = d + 2$ (Proposition 5). Now assume that $Z$ is not reduced, i.e., assume $P \in \tau(X)$. Let $C \subset \mathbb{P}^m$ be a smooth conic containing $A$. The curve $\nu_d(C)$ is a degree 2d rational normal curve in its linear span. Since $P \in \langle Z \rangle \subset \langle \nu_d(C) \rangle$, the “Moreover” part of Proposition 6 applied to $\nu_d(C)$ gives the existence of a set $B \subset C$ such that $\tilde{z}(B) = 2d$ and $(Z) \cap \langle \nu_d(B) \rangle = \{ P \}$. Let $M \subset \mathbb{P}^m$ be the plane containing $C \cup L$. Since the restriction maps $H^0(\mathbb{P}^m, \mathcal{O}_m(d)) \to H^0(M, \mathcal{O}_M(d))$ and $H^0(M, \mathcal{O}_M(d)) \to H^0(T, \mathcal{O}_T(d))$ are surjective for $T = L, T = C$, and $T = C \cup L$, we get $\dim(\langle \nu_d(C) \cup L \rangle) = 3d - 1$, $\dim(\langle \nu_d(C) \rangle) = 2d$ and $\dim(\langle R \rangle) = d$. Hence Grassmann’s formula gives $\langle \nu_d(C) \rangle \cap \langle R \rangle = \langle Z \rangle$. Fix $E \subset L$ such that $\{ P \} = \langle Z \rangle \cap \langle \nu_d(E) \rangle$ (the “Moreover” part of Proposition 6). Since $\nu_d(E) \subset R$, $P$ is the only point in the intersection of $\langle \nu_d(B) \rangle \subset \langle \nu_d(C) \rangle$ and $\langle \nu_d(E) \rangle$. Hence $\alpha(X, P) \leq 3d$. Now assume $a := \alpha(X, P) < 3d$ and take $S = S_1 \cup \cdots \cup S_k \subset \mathbb{P}^m$ such that $\tilde{z}(S) = a$ and $\{ P \} = \bigcap_{i=1}^k \langle \nu_d(S_i) \rangle$. We proved that $\tilde{z}(S_i) \geq d$ for all $i$. Hence $k = 2$, $2d \leq a \leq 3d - 1$ and $d \leq \tilde{z}(S_i) \leq 2d - 1$ for all $i$.

Claim: Take a finite set $E \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(E) \rangle$, $P \notin \langle E' \rangle$ for any $E' \subset E$, $E \neq A$, and $\deg(E) \leq 2d - 1$. Then $E \subset L$.

Proof of the Claim: Since $P \in \langle Z \rangle$, Lemma 3 and [8], Lemma 34, give the existence of a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap (E \cup A)) \geq d + 2$. First we will check that $E \subset D$ and then we will see that $D = L$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing $D$. Since $E$ is reduced, $A$ is curvilinear and $H$ is general, we have $H \cap (A \cup E) = D \cap (A \cup E)$. Let $Res_H(A \cup E)$ denote the residual scheme of $A \cup E$ with respect to $H$, i.e., the closed subscheme of $\mathbb{P}^m$ with $\mathcal{I}_{A \cup E} : \mathcal{I}_H$ as its ideal sheaf. Since $\deg(Res_H(A \cup E)) = \deg(A \cup E) - \deg((A \cup E) \cap H) \leq d$, we have $h^1(\mathbb{P}^m, \mathcal{I}_{Res_H(A \cup E)}(d - 1)) = 0$. Since $A$ is connected and not reduced, [6], Lemma 4, gives $A \cup E \subset H$. Since this is true for a general $H$ containing $D$, we get $E \subset D$. We also get $A \subset D$ and hence $D = L$.

Apply the Claim first to $S_1$ and then to $S_2$. We get $S \subset L$. Hence $\alpha(X, P) = \alpha(R, P) = d^2$, a contradiction. \qed
Theorem 5. Fix a linear subspace $U \subseteq \mathbb{P}^m$ and take $P \in \langle \nu_d(U) \rangle$. We have $r_{x_m,d}(P) = r_{\nu_d(U)}(P)$ ([21], Proposition 3.1) and every $S \subset X$ evincing $r_X(P)$ is contained in $\nu_d(U)$ ([19], Exercise 3.2.2.2). Part (b) of Theorem 4 shows that sometimes $ir_X(P) < ir_{nu_d(U)}(P)$.

Theorem 5. Assume $m \geq 2$ and $d \geq 5$. Fix a finite set $A \subset \mathbb{P}^m$ such that $\sharp(A) = 3$. Set $X := X_{m,d}$ and $n := (m + d) - 1$. Fix $P \in \langle \nu_d(A) \rangle$ such that $P \notin \langle \nu_d(A') \rangle$ for any $A' \subset A$.

(a) Assume that $A$ is contained in a line. Then $r_X(P) = \nu_d(P) = d - 1$ and $\alpha(X, P) = \gamma(X, P) = d + 2$.

(b) Assume that $A$ is not contained in a line. Then $r_X(P) = \nu_d(P) = d - 1$ and $\alpha(X, P) = 2d + 2$.

Proof. Since $\beta(X) \geq 5$, $\nu_d(A)$ is the only subscheme of $X$ with degree $\leq 3$ whose linear span contains $P$. Hence $r_X(P) = \nu_d(P) = d - 1$ and $\alpha(X, P) = \gamma(X, P) = d + 2$.

First assume the existence of a line $L \subset \mathbb{P}^m$ such that $A \subset L$. Set $R := \langle \nu_d(L) \rangle$. Since $P \in \langle R \rangle$, Proposition 7 gives $ir_X(P) \leq ir_R(P) = d - 1$, $i\nu_X(P) \leq i\nu_R(P) = d - 1$, $\alpha(X, P) \leq \alpha(R, P) = d + 2$ and $\gamma(X, P) \leq \gamma(R, P) = d + 2$, concluding the proof of part (a).

Now assume that $A$ is not contained in a line. Write $A = \{O_1, O_2, O_3\}$. Fix $i \in \{1, 2, 3\}$ and set $L_i := \{O_3, O_i\} \subset \mathbb{P}^m$. Since $P \in \langle \nu_d(A) \rangle$ and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subset A$, the set $\langle \{O_3, O_i\} \rangle \cap \langle \nu_d(O_i), \nu_d(O_3) \rangle$ is a single point, $P_i$. Notice that $P_i \in \langle \nu_d(L_i) \rangle$ and that $r_{\nu_d(L_i)}(P_i) = 2$. The “Moreover” part of Proposition 5 gives the existence of a set $E_i \subset L_i$ such that $\sharp(S_i) = d$ and $\langle P_i \rangle = \langle \{O_3, O_i\} \cup \nu_d(E_i) \rangle$. Hence $\langle \nu_d(A) \rangle \cap \langle \nu_d(O_i) \cup \nu_d(E_i) \rangle$ is the line $\langle \nu_d(O_i), P_i \rangle$. Taking the intersection of two of these lines we get $r_X(P) \leq d + 1$ and $\alpha(X, P) \leq 2d + 2$. Since $r_X(P) = d + 1$ (proof of this case in [8], Theorem 37), we get $ir_X(P) = d + 1$. Lemma 3 also gives $i\nu_X(P) \geq d + 1$ and $\gamma(X, P) \geq d + 1$ and that for each subscheme $W \subset \mathbb{P}^m$ with deg($W$) $\leq d + 1$ and $P \in \langle W \rangle$ we have $W \supseteq A$. Hence $i\nu_X(P) = d + 1$. Assume $a := \alpha(X, P) \leq 2d + 1$ and take $S = S_1 \cup \cdots \cup S_k$ with $\langle P \rangle = \bigcap_{i=1}^k \langle \nu_d(S_i) \rangle$ and $\sharp(S_i) = \cdots = \sharp(S_k) = a$. Since $a \leq 2d + 1$ and each subscheme $W \subset \mathbb{P}^m$ with deg($W$) $\leq d + 1$ and $P \in \langle W \rangle$ contains $A$, we get $k = 2$ and that one of the sets $S_i$ is just $A$. Since $P \in \langle S_1 \rangle \cap \langle S_2 \rangle$, $P \notin \langle U \rangle$ for any $U \subset S_i$, $i = 1, 2$, and $\sharp(S_1 \cup S_2) = 2d + 1$, there is a line $D \subset \mathbb{P}^m$ such that $\sharp(D \cap (S_1 \cup S_2)) \geq d + 2$ and $S_1 \setminus S_1 \cap D = S_2 \setminus S_2 \cap D$ ([6], Lemma 4). Since $S_1 \cap S_2 = \emptyset$, we get $S_1 \cup S_2 \supset D$. Since $A$ is not contained in a line and $A = S_i$ for some $i$, we get a contradiction.

References