ALBANIAN JOURNAL
OF MATHEMATICS
Volume 1, Number 1, Pages 13–16
ISSN 1930-1235(electronic version)

THE SKOLEM PROBLEM FOR 2×2 MATRICES, ARCTANGENTS AND RECURSIVE SOLVABILITY

JODY M. LOCKHART

(Communicated by D. Joyner)

ABSTRACT. A new short proof of the solvability of the Skolem problem for two by two matrices is given.

1. Introduction

Decision problems for groups have been studied very extensively starting in the 1950's. More recently, decision problems for finite sets of matrices have been investigated. In 1970, Paterson [5] showed that the mortality problem is unsolvable; a set of $n \times n$ matrices is said to be *mortal* if some finite product of elements in the set is the zero matrix and the mortality problem is the problem of deciding if finite sets of $n \times n$ matrices with integer entries are mortal.

The problem that we consider is the Skolem problem. The Skolem problem is the problem of deciding for a given square matrix with integer entries whether there is some positive power of the matrix that has zero as its entry in the upper right corner. In 1997, V. Halava [3] gave a proof of the solvability of the Skolem problem for 2×2 matrices. The Skolem problem for 3×3 and 4×4 matrices was solved in 1985 by N. K. Vereshchagin [7] and the problem for 5×5 matrices was solved in 2005 by V. Halava, T. Harju, M. Hirvensalo, and J. Karhumaki [4]. In this note, we give a short new proof for the 2×2 case that ties the Skolem problem to a problem about arctangents and uses a beautiful result of J. H. Conway, C. Radin, and L. Sadun [2] about geodetic angles.

2. Skolem and Arctangent Problems

In this section, we first show that the Skolem problem for 2×2 matrices with integer entries and with real eigenvalues is solvable. Then we show that the Skolem problem for 2×2 matrices with integer entries and with non-real eigenvalues reduces to the following problem about arctangents.

Arctangent Problem: Given positive integers m and n, is $\arctan(\frac{\sqrt{m}}{n})$ a rational multiple of π?

Received by the editors December 1, 2006 and, in revised form, February 10, 2007.
2000 *Mathematics Subject Classification.* Primary 15A36.
Key words and phrases. Skolem problem, arctangents.

Proposition 1. *A 2×2 matrix A of integers with real eigenvalues has a positive integer power with upper right corner zero if and only if either A or A^2 has upper right corner zero.*

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $a_{ij} \in \mathbb{Z}$ for $i, j = 1, 2$. If A has two distinct real eigenvalues λ_1 and λ_2, then there is a nonsingular matrix $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ such that $B^{-1}AB = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Then, for any positive integer n,

$$\begin{aligned} A^n &= B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n B^{-1} \\ &= B \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} B^{-1} \\ &= \begin{bmatrix} * & \frac{b_{11}b_{12}}{d}(\lambda_2^n - \lambda_1^n) \\ * & * \end{bmatrix}, \end{aligned} \tag{1}$$

where d is the determinant of B. If $b_{11}b_{12} = 0$, then the upper right corner of A^1 is zero and we are done. Otherwise, the upper right corner of A^n is zero if and only if $\lambda_1^n = \lambda_2^n$. Since λ_1 and λ_2 are real numbers, $\lambda_1^n = \lambda_2^n$ for some positive integer n if and only if $\lambda_1 = \pm\lambda_2$ if and only if $\lambda_1^2 = \lambda_2^2$ if and only if the upper right corner of A^2 is zero.

Next, suppose that A has one real eigenvalue λ. Then there is a nonsingular matrix $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ such that either

$$B^{-1}AB = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad B^{-1}AB = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

If $B^{-1}AB = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ and A^1 has upper right corner zero. If $B^{-1}AB = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$, then

$$\begin{aligned} A^n &= B \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}^n B^{-1} \\ &= B \begin{bmatrix} \lambda^n & 0 \\ n\lambda^{n-1} & \lambda^n \end{bmatrix} B^{-1} \\ &= \frac{1}{d} \begin{bmatrix} * & -b_{12}^2 n\lambda^{n-1} \\ * & * \end{bmatrix}, \end{aligned} \tag{2}$$

and the upper right corner of A^n is zero if and only if $b_{12} = 0$ or $\lambda = 0$. If $b_{12} = 0$ then the upper right corner of A^1 is zero, and if $\lambda = 0$ then the upper right corner of A^2 is zero.

□

Corollary 2. *The Skolem problem for 2×2 matrices with integer coefficients and with real eigenvalues is solvable.*

Next, consider the case of non-real eigenvalues.

Proposition 3. *The Skolem problem for 2×2 matrices with integer coefficients and with non-real eigenvalues reduces to the arctangent problem.*

Proof. Let A be a 2×2 matrix as above and let λ_1 and λ_2 be its eigenvalues. Since λ_1 and λ_2 are non-real conjugates, they are unequal and as in Eq. (1) of the proposition above, we get

$$A^n = \begin{bmatrix} * & \frac{b_{11}b_{12}}{d}(\lambda_2^n - \lambda_1^n) \\ * & * \end{bmatrix},$$

where $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and $B^{-1}AB = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. As above, if $b_{11}b_{12} = 0$, the upper right corner of A^1 is 0. If $b_{11}b_{12} \neq 0$, then the upper right corner of A^n is zero if and only if $\lambda_1^n = \lambda_2^n$. Therefore, the Skolem problem reduces to the problem of determining the existence of a positive integer n such that $\lambda_1^n = \lambda_2^n$.

Computing the eigenvalues of A, we get

$$\begin{aligned} \lambda_j &= \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})}}{2}, \text{ for } j = 1, 2 \\ &= \frac{\ell \pm i\sqrt{m}}{2}, \text{ where } \ell, m \in \mathbb{Z} \text{ and } m > 0. \end{aligned}$$

Let $\theta = \arctan(\frac{\sqrt{m}}{\ell})$ and $r = \frac{1}{2}\sqrt{\ell^2 + m}$. Since $\lambda_1 = r\,e^{i\theta}$ and $\lambda_2 = r\,e^{-i\theta}$,

$$\lambda_1^n = \lambda_2^n \Leftrightarrow e^{in\theta} = e^{-in\theta} \Leftrightarrow e^{2in\theta} = 1 \Leftrightarrow n\theta = \pi t,$$

for some $t \in \mathbb{Z}$. So there is a positive integer n such that the upper right corner of A^n is zero if and only if there is a positive integer n and an integer $t \not\equiv 0 (\operatorname{mod} n)$ such that $\theta = \frac{t}{n}\pi$. Therefore, there is a positive integer n such that the upper right corner of A^n is zero if and only if $\arctan(\frac{\sqrt{m}}{\ell}) \in \pi(\mathbb{Q} - \mathbb{Z})$. Since we know that $\arctan(\frac{\sqrt{m}}{\ell}) \notin \pi\mathbb{Z}$, the problem reduces to that of deciding whether or not $\arctan(\frac{\sqrt{m}}{\ell}) \in \pi\mathbb{Q}$.

□

3. Solvability of Arctangent and Skolem Problems

In this section, we will see that the arctangent problem is solvable and thus the Skolem problem for 2×2 matrices over $\mathbb{Z}$ is also solvable. That the arctangent problem is solvable follows from a result of J. H. Conway, C. Radin, and L. Sadun [2]. In [2], they define a *pure geodetic angle* θ to be an angle such that "any one (and therefore each) of its six squared trigonometric functions is rational (or infinite)." The angles $\arctan(\frac{\sqrt{m}}{\ell})$ that we are interested in are pure geodetic angles. Since $\arctan(-\alpha) = -\arctan\alpha$, we may assume that ℓ is positive. Conway et al [2] define angles $\langle p \rangle_d$ satisfying the following conditions.

Condition 1: **Theorem** (Conway, Radin and Sadun) Every pure geodetic angle is uniquely expressible as a rational multiple of π plus an integral linear combination of the angles $\langle p \rangle_d$.

Condition 2: Rewrite $\frac{\sqrt{m}}{\ell}$ as $\frac{b\sqrt{d}}{a}$ where $a, b, d \in \mathbb{Z}^+$, $\gcd(a, b) = 1$, and d is square free. Then the only $\langle p \rangle_d$ that occur in the expression for $\arctan(\frac{b\sqrt{d}}{a})$ are those for which p is a prime divisor of $a^2 + db^2$ and for which the ideal (p) splits in

$\mathcal{O}_d$, the ring of integers of $\mathbb{Q}(\sqrt{-d})$.

If any combination of the $\langle p\rangle_d$ were a rational multiple of π, the expression would not be unique. Therefore, θ is a rational multiple of π if and only if no $\langle p\rangle_d$ occurs in its expression. So, to determine whether $\arctan(\frac{\sqrt{m}}{\ell})$ is a rational multiple of π, we need to determine whether its expression contains any $\langle p\rangle_d$.

Recall that an ideal I is *prime* if $xy \in I$ implies that $x \in I$ or $y \in I$. By definition, an ideal (p) *splits* if $(p) = P_1P_2...P_k$, where the P_i are distinct prime ideals and $k > 1$. An ideal (p) *ramifies* if the factorization of (p) contains a repeated prime ideal. Thus, an ideal splits if and only if it is not prime and is unramified. It is known that if p is an odd prime, then (p) ramifies in $\mathcal{O}_d$ if and only if p divides d ([6], p. 101) and (p) is prime in $\mathcal{O}_d$ if and only if $-d$ is not a nonzero square modulo p ([2], Theorem 5, p. 329). For $p = 2$, the ideal (p) splits in $\mathcal{O}_d$ if and only if $d \equiv 7 (\mathrm{mod}\, 8)$ ([2], Theorem 6, p. 329) . Therefore, we have the following results.

Theorem 4. *Let a, b, and d be positive integers for which $\gcd(a, b) = 1$ and d is square free. Then $\arctan(\frac{b\sqrt{d}}{a})$ is a rational multiple of π if and only if all prime factors p of $a^2 + db^2$ satisfy the following conditions.*

(i) If p is an odd prime then either $p|d$ or $-d$ is not a nonzero square modulo p.
(ii) If $p = 2$, then $d \neq 7(\mathrm{mod}\, 8)$.

Corollary 5. *There is an algorithm which, given positive integers ℓ and m determines whether or not $\arctan(\frac{\sqrt{m}}{\ell})$ is a rational multiple of π.*

Corollary 6. *The Skolem problem for 2×2 matrices of integers is solvable.*

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Department of Mathematics,, U. S. Naval Academy,, Annapolis,Maryland
E-mail address: jml@usna.edu