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COMMON FIXED POINT THEOREMS OF GENERALIZED CONTRACTION, ZAMFIRESCU PAIR OF MAPS IN CONE METRIC SPACES

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Abstract. We prove the existence of common fixed points of a generalized contraction / Zamfirescu pair of maps in a complete cone metric space. Our results generalize the results of Huang anf Zhang [L-G. Huang, X. Zhang: Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476] and extend the results of Rezapour and Hamlbarani [Sh. Rezapour, R. Hamlbarani: Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl. 345 (2008) 719–724].

1. Introduction

In 2007, Huang and Zhang [2] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space by defining the concept of a cone metric space which is more general than that of a metric space, and obtained fixed point theorems for mappings on complete cone metric spaces having normal cone. Later in 2008, Rezapour and Hamlbarani [4] generalized the results of Huang and Zhang [2] by relaxing the normality property of the cones.

Recently, Jungck, Radenović, Radojević and Rakočević [3] have studied common fixed point results for weakly compatible pairs of mappings in cone metric spaces by omitting the assumptions of normality of the cone in their results, which generalize and extend some earlier results ([2], [4], [1],[5]).

Thoughout this paper we use the following notation: $\mathbb{R}$ denotes the set of all reals; and $\mathbb{N}$ denotes the set of all natural numbers.

Let E be a real Banach space and P be a subset of E. P is called a *cone* if the following three conditions hold:

(1) P is closed, nonempty, and $P \neq \{0\}$,
(2) $a, b \in \mathbb{R}$, $a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$, and
(3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial order $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. In this case we call P an order cone. We write $x < y$ if $x \leq y$ and $x \neq y$; we write $x \ll y$ if $y - x \in int\ P$, where $int\ P$ denotes the interior of P.

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An order cone P is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \textit{ implies } \parallel x \parallel \leq K \parallel y \parallel .$$

The least positive number satisfying the above inequality is called the normal constant of P.

Rezapour and Hamlbarani [4] observed that there is no normal cone with normal constant $K < 1$. There exists an ordered Banach space E with cone P which is not normal but $\text{int}P \neq \emptyset$.

Definition 1.1. Let X be a nonempty set. If the mapping $d : X \times X \to E$ satisfies

(1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$, and
(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$,

then d is called a *cone metric* on X, and (X, d) is called *a cone metric space.*

Definition 1.2. Let (X, d) be a cone metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

(1) a Cauchy sequence in X if for each c in E with $0 \ll c$, there is an N such that for all $m, n > N$, $d(x_m, x_n) \ll c$;
(2) a convergent sequence in X if for each c in E with $0 \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$ for some x in X. In this case, we say that $\{x_n\}$ converges to x in X and we denote it by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

A cone metric space is said to be *complete* if every Cauchy sequence in X is convergent in X.

Remark 1.3. [3] Let E be an ordered Banach (normed) space with a cone P.

(1) c is an interior point of the cone P if and only if $[-c, c]$ is a neighborhood of 0.
(2) If $a \leq b$ and $b \ll c$, then $a \ll c$.
(3) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(4) If $0 \leq u \ll c$ for each $c \in \text{int}P$, then $u = 0$.
(5) If $c \in \text{int}P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.
(6) Let $0 \ll c$. If $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then eventually $d(x_n, x) \ll c$, where $\{x_n\}$ is a sequence in X and x is a given point in X.
(7) If $a_n \leq b_n$ and $a_n \to a$, $b_n \to b$, then $a \leq b$ for each cone P.
(8) If E is a real Banach space with cone P and $a \leq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

From Remark 1.3 (4) and (5), we observe that if a sequence $\{x_n\}$ is convergent in a cone metric space with a cone P having nonempty interior, then the limit of $\{x_n\}$ is unique.

Definition 1.4. [3] Let (X, d) be a cone metric space and P a cone with nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that $f(X) \subset g(X)$, and $f(X)$ or $g(X)$ is a complete subspace of X. Then the pair (f, g) is called Abbas and Jungck's pair, or shortly AJ's pair.

Definition 1.5. [3]. Let f and g be selfmaps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. f and g are said to be weakly compatible if they commute at their coincidence point; that is, if $fx = gx$ for some x in X, then $fgx = gfx$.

Recently, Jungck *et. al.* [3] proved the following theorems.

Theorem 1.6. (Jungck *et. al.* [3], Theorem 2.1). *Suppose that (f, g) is AJ's pair, and that for some constant $k \in (0, 1)$ and for every $x, y \in X$, there exists*

$$p(x,y) \in \{d(gx,gy), d(fx,gx), d(fy,gy), \frac{d(fx,gy)+d(fy,gx)}{2}\}, \tag{1.6.1}$$

such that

$$d(fx,fy) \leq k\ p(x,y). \tag{1.6.2}$$

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Theorem 1.7. (Jungck *et. al.* [3], Theorem 2.2). *Suppose that (f, g) is AJ's pair, and that for some constant $k \in (0, 1)$ and for every $x, y \in X$, there exists*

$$p(x,y) \in \{d(gx,gy), \frac{d(fx,gx)+d(fy,gy)}{2}, \frac{d(fx,gy)+d(fy,gx)}{2}\}, \tag{1.7.1}$$

such that

$$d(fx,fy) \leq k\ p(x,y). \tag{1.7.2}$$

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

We now introduce a generalized contraction pair of mappings.

Definition 1.8. Let (X, d) be a cone metric space and P a cone with nonempty interior. Let $f, g : X \to X$ be selfmaps. Suppose that there exists a constant $k \in (0, 1)$ and there exists

$$p(x,y) \in \{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}\}, \tag{1.8.1}$$

such that

$$d(fx,gy) \leq k\ p(x,y) \text{ for all } x,y \text{ in } X. \tag{1.8.2}$$

Then the pair (f, g) is called a generalized contraction pair on X.

The following examples, Example 1.9 and Example 1.10, show that a pair of maps that satisfies inequality (1.6.2) and a generalized contraction pair are independent.

Example 1.9. Let $X = [0, 1]$, $E = C^1_{\mathbb{R}}[0, 1]$ and $P = \{\varphi \in E : \varphi \geq 0\}$. Then P is a cone with nonempty interior. We observe that P is not normal [4].
We define $d : X \times X \to E$ by $d(x, y) = |x - y|\varphi$, where $\varphi : [0, 1] \to \mathbb{R}$ by $\varphi(t) = e^t$. Then d is a cone metric on X. We define mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{3}x & \text{if } 0 \leq x < \frac{5}{6} \\ \frac{1}{3} & \text{if } \frac{5}{6} \leq x \leq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{5}{6} \\ \frac{1}{3}x & \text{if } \frac{5}{6} \leq x \leq 1. \end{cases}$$

We observe that the pair (f, g) is a generalized contraction pair with $k = \frac{4}{5}$.

We also observe that neither $f(X) \subset g(X)$ nor $g(X) \subset f(X)$. Hence the pair (f, g) is not AJ's pair. Also, f and g do not satisfy the inequality (1.6.2). For we choose $x = 0$ and $y = 1$. Then for all $k \in (0, 1)$ we have

$$\frac{1}{3}\varphi = d(f0, f1) > k\ p(0, 1),$$

where

$$p(0,1) \in \{d(g0, g1), d(f0, g0), d(f1, g1), \frac{d(f0, g1) + d(f1, g0)}{2}\} = \{0, \frac{1}{3}\varphi\};$$

and

$$\frac{1}{3}\varphi = d(g0, g1) > k\ p(0, 1),$$

where

$$p(0,1) \in \{d(f0, f1), d(g0, f0), d(g1, f1), \frac{d(g0, f1) + d(g1, f0)}{2}\} = \{0, \frac{1}{3}\varphi\}.$$

Example 1.10. Let $X = \mathbb{R}$, $E = C^1_{\mathbb{R}}[0, 1]$ and $P = \{\varphi \in E : \varphi \geq 0\}$.
We define $d : X \times X \to E$ by $d(x, y) = |x - y|\varphi$, where $\varphi : [0, 1] \to \mathbb{R}$ by $\varphi(t) = e^t$. Then d is a cone metric on X. We define mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{3}x - 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x - 2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then f and g satisfy the inequality (1.6.2) with $k = \frac{1}{2}$. However, the pair (f, g) is not a generalized contraction pair. For, we choose $x = 2$ and $y = -1$. Then, for all $k \in (0, 1)$, we have

$$3\varphi = d(f2, g(-1)) > k\ p(2, -1),$$

where

$$p(2, -1) \in \{d(2, -1), d(2, f2), d(-1, g(-1)), \frac{d(2, g(-1)) + d(-1, f2)}{2}\} = \{2\varphi, 3\varphi\}.$$

We now introduce a Zamfirescu pair in a cone metric space.

Definition 1.11. Let (X, d) be a cone metric space and P a cone with nonempty interior. Let $f, g : X \to X$ be selfmaps. Suppose that there exists a constant $k \in (0, 1)$ and there exists

$$p(x, y) \in \{d(x, y), \frac{d(x, fx) + d(y, gy)}{2}, \frac{d(x, gy) + d(y, fx)}{2}\}, \tag{1.11.1}$$

such that

$$d(fx, gy) \leq k\ p(x, y) \text{ for all } x, y \text{ in } X. \tag{1.11.2}$$

Then the pair (f, g) is called a Zamfirescu pair on X.

The following examples, Example 1.12 and Example 1.13, show that a pair of maps that satisfies inequality (1.7.2) and a Zamfirescu pair are independent.

Example 1.12. Let X, E, P, d and φ be as in Example 1.9.

We define mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{4}x & \text{if } x \neq 1 \\ \frac{1}{5} & \text{if } x = 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{5}x & \text{if } x \neq 1 \\ \frac{1}{4} & \text{if } x = 1. \end{cases}$$

We observe that the pair (f,g) is a Zamfirescu pair with $k = \frac{2}{3}$.

We also observe that neither $f(X) \subset g(X)$ nor $g(X) \subset f(X)$. Hence the pair (f,g) is not AJ's pair. Also, f and g do not satisfy the inequality (1.7.2). For, by choosing $x = 0$ and $y = \frac{1}{2}$, and for all $k \in (0,1)$ we obtain

$$\frac{1}{8}\varphi = d(f0, f\frac{1}{2}) > k\ p(0, \frac{1}{2}),$$

where

$$p(0,\frac{1}{2}) \in \{d(g0, g\frac{1}{2}), \frac{d(f0,g0)+d(f\frac{1}{2},g\frac{1}{2})}{2}, \frac{d(f0,g\frac{1}{2})+d(f\frac{1}{2},g0)}{2}\} = \{\frac{1}{10}\varphi, \frac{1}{80}\varphi, \frac{9}{80}\varphi\},$$

Now, by taking $x = 0$ and $y = 1$, for all $k \in (0,1)$ we obtain

$$\frac{1}{4}\varphi = d(g0, g1) > k\ p(0,1),$$

where

$$p(0,1) \in \{d(f0,f1), \frac{d(g0,f0)+d(g1,f1)}{2}, \frac{d(g0,f1)+d(g1,f0)}{2}\} = \{\frac{1}{5}\varphi, \frac{1}{40}\varphi, \frac{9}{40}\varphi\}.$$

Example 1.13. Let X, E, P, d, φ, f and g be as in Example 1.10.
Then f and g satisfy the inequality (1.7.2) with $k = \frac{1}{2}$. However, the pair (f,g) is not a Zamfirescu pair. For, we choose $x = 2$ and $y = -1$. Then, for all $k \in (0,1)$, we have

$$3\varphi = d(f2, g(-1)) > k\ p(2,-1),$$

where

$$\begin{aligned} p(2,-1) &\in \{d(2,-1), \frac{d(2,f2)+d(-1,g(-1))}{2}, \frac{d(2,g(-1))+d(-1,f2)}{2}\} \\ &= \{2\varphi, \frac{5}{2}\varphi, 3\varphi\}. \end{aligned}$$

The aim of this paper is to prove the existence of common fixed points of a generalized contraction pair in a complete cone metric space. The same is also established for Zamfirescu pair in Section 3. Our results generalize the results of Huang and Zhang [2] and extend the results of Rezapour and Hamlbarani [4].

2. Common fixed point theorems of a generalized contraction pair

Theorem 2.1. *Let (X,d) be a complete cone metric space. Suppose that (f,g) is a generalized contraction pair on X. Then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. Since $f(X) \subset X$, there exists $x_1 \in X$ such that $x_1 = fx_0$. Since $g(X) \subset X$, there exists $x_2 \in X$ such that $x_2 = gx_1$. By continuing this process, having defined $x_n \in X$, we define $x_{n+1} \in X$ such that

$$x_{n+1} = \begin{cases} fx_n & \text{if } n = 0,2,4,\cdots \\ gx_n & \text{if } n = 1,3,5,\cdots. \end{cases}$$

We first show that

$$d(x_{n+1},x_n) \le k\ d(x_n,x_{n-1}),\ \text{for } n = 1,2,3,\cdots. \tag{2.1.1}$$

We consider two cases.
Case (i): n is even. Then,

$$d(x_{n+1},x_n) = d(fx_n, gx_{n-1}) \le k\ p(x_n,x_{n-1}),$$

where

$$p(x_n, x_{n-1}) \in \{d(x_n, x_{n-1}), d(x_n, fx_n), d(x_{n-1}, gx_{n-1}), \frac{d(x_n, gx_{n-1}) + d(x_{n-1}, fx_n)}{2}\}$$

$$= \{d(x_n, x_{n-1}), d(x_{n+1}, x_n), \frac{1}{2}d(x_{n+1}, x_{n-1})\}.$$

Now if $p(x_n, x_{n-1}) = d(x_n, x_{n-1})$, then clearly (2.1.1) holds;
if $p(x_n, x_{n-1}) = d(x_{n+1}, x_n)$, then from Remark 1.3 (8), we have $d(x_{n+1}, x_n) = 0$, and hence (2.1.1) holds;
if $p(x_n, x_{n-1}) = \frac{1}{2}d(x_{n+1}, x_{n-1})$, then we have

$$d(x_{n+1}, x_n) \leq \frac{1}{2}d(x_{n+1}, x_n) + \frac{k}{2}d(x_n, x_{n-1}),$$

and hence (2.1.1) holds.
Case (ii): n is odd. Then,

$$d(x_{n+1}, x_n) = d(gx_n, fx_{n-1}) = d(fx_{n-1}, gx_n) \leq k\ p(x_{n-1}, x_n),$$

where

$$p(x_{n-1}, x_n) \in \{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, gx_n), \frac{d(x_{n-1}, gx_n) + d(x_n, fx_{n-1})}{2}\}$$

$$= \{d(x_n, x_{n-1}), d(x_{n+1}, x_n), \frac{1}{2}d(x_{n+1}, x_{n-1})\}.$$

Now if $p(x_{n-1}, x_n) = d(x_n, x_{n-1})$, then clearly (2.1.1) holds;
if $p(x_{n-1}, x_n) = d(x_{n+1}, x_n)$, then from Remark 1.3 (8), we have $d(x_{n+1}, x_n) = 0$, and hence (2.1.1) holds;
if $p(x_{n-1}, x_n) = \frac{1}{2}d(x_{n+1}, x_{n-1})$, then we have

$$d(x_{n+1}, x_n) \leq \frac{1}{2}d(x_{n+1}, x_n) + \frac{k}{2}d(x_n, x_{n-1}),$$

and hence (2.1.1) holds.

Hence, in both cases the inequality (2.1.1) holds.

By repeated application of (2.1.1), we get

$$d(x_{n+1}, x_n) \leq k^n\ d(x_1, x_0),\ n = 1, 2, \cdots. \tag{2.1.2}$$

We show that $\{x_n\}$ is a Cauchy sequence in X. For $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq (k^{n-1} + k^{n-2} + \cdots + k^m)d(x_1, x_0) \\ &\leq \frac{k^m}{1-k}d(x_1, x_0) \to 0 \text{ as } m \to \infty. \end{aligned} \tag{2.1.3}$$

Let $0 \ll c$. From (2.1.3) and Remark 1.3 (5), there exists an integer N such that $k^m(1-k)^{-1}d(x_1, x_0) \ll c$ for all $m > N$. By Remark 1.3 (2), $d(x_n, x_m) \ll c$. Hence, by Definition 1.2 (1), $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists z in X such that $x_n \to z$ as $n \to \infty$.

We claim that $fz = z$.

Let $0 \ll c$. Without loss of generality we assume that n is odd. Then,

$$\begin{aligned} d(fz, z) &\leq d(fz, gx_n) + d(gx_n, z) \\ &\leq k\ p(z, x_n) + d(x_{n+1}, z), \end{aligned} \tag{2.1.4}$$

where

$$p(z, x_n) \in \{d(z, x_n), d(z, fz), d(x_n, gx_n), \frac{d(z, gx_n) + d(x_n, fz)}{2}\}$$

$$= \{d(z, x_n), d(z, fz), d(x_n, x_{n+1}), \frac{d(z, x_{n+1}) + d(x_n, fz)}{2}\}.$$

Clearly one of the following cases hold for infinitely many n.
If $p(z, x_n) = d(z, x_n)$, then from (2.1.4) we have

$$d(fz, z) \leq k\ d(z, x_n) + d(x_{n+1}, z) \ll k\ \frac{c}{2k} + \frac{c}{2} = c;$$

if $p(z, x_n) = d(z, fz)$, then from (2.1.4) we get

$$d(fz, z) \leq \frac{1}{1-k}\ d(z, x_{n+1}) \ll \frac{1}{1-k}\ \frac{c}{\frac{1}{1-k}} = c;$$

if $p(z, x_n) = d(x_n, x_{n+1})$, then from (2.1.4) we get

$$d(fz, z) \leq k\ d(x_n, z) + (1+k)\ d(x_{n+1}, z) \ll k\ \frac{c}{2k} + (1+k)\frac{c}{2(1+k)} = c;$$

if $p(z, x_n) = \frac{d(z, x_{n+1}) + d(x_n, fz)}{2}$, then from (2.1.4) we get

$$d(fz, z) \leq k\ \frac{d(z, x_{n+1}) + d(x_n, fz)}{2} + d(x_{n+1}, z)$$

$$\leq (1 + \frac{k}{2})\ d(z, x_{n+1}) + \frac{k}{2}\ d(x_n, z) + \frac{1}{2} d(z, fz)$$

so that

$$d(fz, z) \leq (2+k)\ d(z, x_{n+1}) + k\ d(x_n, z) \ll (2+k)\ \frac{c}{2(2+k)} + k\frac{c}{2k} = c.$$

In all cases, we obtain $d(fz, z) \ll c$ for each $c \in \operatorname{int} P$. Using Remark 1.3 (4), it follows that $d(fz, z) = 0$, or $fz = z$.

Next we prove that $gz = z$.
Consider

$$d(z, gz) = d(fz, gz) \leq k\ p(z, z), \tag{2.1.5}$$

where

$$p(z, z) \in \{d(z, z), d(z, fz), d(z, gz), \frac{d(z, fz) + d(z, gz)}{2}\}$$

$$= \{0, d(z, gz), \frac{d(z, gz)}{2}\}.$$

Now, if $p(z, z) = 0$, from (2.1.5) trivially we get $gz = z$. If either $p(z, z) = \frac{d(z,gz)}{2}$ or $p(z, z) = d(z, gz)$, then from (2.1.5) and Remark 1.3 (8), we have $d(z, gz) = 0$; i.e., $z = gz$.

Hence, $fz = gz = z$.

The uniqueness of z follows from the inequality (1.8.2). Hence the theorem follows. □

The following is an example in support of Theorem 2.1.

Example 2.2. Let X, E, P, d, φ, f and g be as in Example 1.9.

The pair (f, g) is a generalized contraction pair with $k = \frac{4}{5}$; and the maps f and g satisfy all the conditions of Theorem 2.1 and 0 is the unique common fixed point of f and g.

3. Common fixed point theorems of Zamfirescu pair

In the following theorem, we prove a common fixed point theorem in cone metric spaces which is an analog of the well-known Zamfirescu result in metric spaces [6].

Theorem 3.1. *Let (X, d) be a complete cone metric space. Suppose that (f, g) is a Zamfirescu pair on X. Then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. Since $f(X) \subset X$, there exists $x_1 \in X$ such that $x_1 = fx_0$. Since $g(X) \subset X$, there exists $x_2 \in X$ such that $x_2 = gx_1$. By continuing this process, having defined $x_n \in X$, we can define $x_{n+1} \in X$ such that

$$x_{n+1} = \begin{cases} fx_n & if\ n = 0, 2, 4, \cdots \\ gx_n & if\ n = 1, 3, 5, \cdots . \end{cases}$$

We first show that

$$d(x_{n+1}, x_n) \leq k\ d(x_n, x_{n-1}), \text{ for } n = 1, 2, 3, \cdots . \tag{3.1.1}$$

We consider two cases.

Case (i): n is even. Then,

$$d(x_{n+1}, x_n) = d(fx_n, gx_{n-1}) \leq k\ p(x_n, x_{n-1}),$$

where

$$p(x_n, x_{n-1}) \in \{d(x_n, x_{n-1}), \frac{d(x_n, fx_n) + d(x_{n-1}, gx_{n-1})}{2}, \frac{d(x_n, gx_{n-1}) + d(x_{n-1}, fx_n)}{2}\}$$
$$= \{d(x_n, x_{n-1}), \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2}, \frac{1}{2}d(x_{n+1}, x_{n-1})\}.$$

Now if $p(x_n, x_{n-1}) = d(x_n, x_{n-1})$, then clearly (3.1.1) holds;
if $p(x_n, x_{n-1}) = \frac{d(x_{n+1},x_n)+d(x_n,x_{n-1})}{2}$, then we have

$$d(x_{n+1}, x_n) \leq k\ \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2}$$
$$\leq \frac{1}{2}d(x_{n+1}, x_n) + \frac{k}{2}d(x_n, x_{n-1}),$$

and hence (3.1.1) holds;
if $p(x_n, x_{n-1}) = \frac{1}{2}d(x_{n+1}, x_{n-1})$, then we have

$$d(x_{n+1}, x_n) \leq \frac{1}{2}d(x_{n+1}, x_n) + \frac{k}{2}d(x_n, x_{n-1}),$$

and hence (3.1.1) holds.

Case (ii): n is odd. Then,

$$d(x_{n+1}, x_n) = d(gx_n, fx_{n-1}) = d(fx_{n-1}, gx_n) \leq k\ p(x_{n-1}, x_n),$$

where

$$p(x_{n-1}, x_n) \in \{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1}) + d(x_n, gx_n)}{2}, \frac{d(x_{n-1}, gx_n) + d(x_n, fx_{n-1})}{2}\}$$

$$= \{d(x_n, x_{n-1}), \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2}, \frac{1}{2}d(x_{n+1}, x_{n-1})\}.$$

Now if $p(x_{n-1}, x_n) = d(x_n, x_{n-1})$, then clearly (3.1.1) holds;
if $p(x_{n-1}, x_n) = \frac{d(x_{n+1},x_n)+d(x_n,x_{n-1})}{2}$, then we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq k\ \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2} \\ &\leq \frac{1}{2}d(x_{n+1}, x_n) + \frac{k}{2}d(x_n, x_{n-1}), \end{aligned}$$

and hence (3.1.1) holds;
if $p(x_{n-1}, x_n) = \frac{1}{2}d(x_{n+1}, x_{n-1})$, then we have

$$d(x_{n+1}, x_n) \leq \frac{1}{2}d(x_{n+1}, x_n) + \frac{k}{2}d(x_n, x_{n-1}),$$

and hence (3.1.1) holds.

Hence, in both cases the inequality (3.1.1) holds.

By repeated application of (3.1.1), we get

$$d(x_{n+1}, x_n) \leq k^n\ d(x_1, x_0),\ n = 1, 2, \cdots . \tag{3.1.2}$$

We show that $\{x_n\}$ is a Cauchy sequence in X. For $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq (k^{n-1} + k^{n-2} + \cdots + k^m)d(x_1, x_0) \\ &\leq \frac{k^m}{1-k}d(x_1, x_0) \to 0 \text{ as } m \to \infty. \end{aligned} \tag{3.1.3}$$

Let $0 \ll c$. From (3.1.3) and Remark 1.3 (5), there exists an integer N such that $k^m(1-k)^{-1}d(x_1, x_0) \ll c$ for all $m > N$. By Remark 1.3 (2), $d(x_n, x_m) \ll c$. Hence, by Definition 1.2 (1), $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists z in X such that $x_n \to z$ as $n \to \infty$.

We claim that $fz = z$.

Let $0 \ll c$. Without loss of generality we assume that n is odd. Then,

$$\begin{aligned} d(fz, z) &\leq d(fz, gx_n) + d(gx_n, z) \\ &\leq k\ p(z, x_n) + d(x_{n+1}, z), \end{aligned} \tag{3.1.4}$$

where

$$\begin{aligned} p(z, x_n) &\in \{d(z, x_n), \frac{d(z, fz) + d(x_n, gx_n)}{2}, \frac{d(z, gx_n) + d(x_n, fz)}{2}\} \\ &= \{d(z, x_n), \frac{d(z, fz) + d(x_n, x_{n+1})}{2}, \frac{d(z, x_{n+1}) + d(x_n, fz)}{2}\}. \end{aligned}$$

Clearly one of the following cases hold for infinitely many n.
If $p(z, x_n) = d(z, x_n)$, then from (3.1.4) we have

$$d(fz, z) \leq k\ d(z, x_n) + d(x_{n+1}, z) \ll k\ \frac{c}{2k} + \frac{c}{2} = c;$$

if $p(z, x_n) = \frac{d(z,fz)+d(x_n,x_{n+1})}{2}$, then from (3.1.4) we get

$$\begin{aligned} d(fz, z) &\leq k\ \frac{d(z, fz) + d(x_n, x_{n+1})}{2} + d(x_{n+1}, z) \\ &\leq (1 + \frac{k}{2})\ d(z, x_{n+1}) + \frac{k}{2}\ d(x_n, z) + \frac{1}{2}d(z, fz), \end{aligned}$$

so that

$$d(fz,z) \leq (2+k)\ d(z,x_{n+1}) + k\ d(x_n,z) \ll (2+k)\ \frac{c}{2(2+k)} + k\frac{c}{2k} = c.$$

if $p(z,x_n) = \frac{d(z,x_{n+1})+d(x_n,fz)}{2}$, then from (3.1.4) we get

$$\begin{aligned} d(fz,z) &\leq k\ \frac{d(z,x_{n+1})+d(x_n,fz)}{2} + d(x_{n+1},z) \\ &\leq (1+\frac{k}{2})\ d(z,x_{n+1}) + \frac{k}{2}\ d(x_n,z) + \frac{1}{2}d(z,fz), \end{aligned}$$

so that

$$d(fz,z) \leq (2+k)\ d(z,x_{n+1}) + k\ d(x_n,z) \ll (2+k)\ \frac{c}{2(2+k)} + k\frac{c}{2k} = c.$$

In all cases, we obtain $d(fz,z) \ll c$ for each $c \in \text{int}\ P$. Using Remark 1.3 (4), it follows that $d(fz,z) = 0$, or $fz = z$.

Next we prove that $gz = z$.

Consider

$$d(z,gz) = d(fz,gz) \leq k\ p(z,z), \tag{3.1.5}$$

where

$$\begin{aligned} p(z,z) &\in \{d(z,z), \frac{d(z,gz)+d(z,fz)}{2}, \frac{d(z,fz)+d(z,gz)}{2}\} \\ &= \{0, \frac{d(z,gz)}{2}\}. \end{aligned}$$

Now if $p(z,z) = 0$, from (3.1.5) trivially we get $gz = z$. If $p(z,z) = \frac{d(z,gz)}{2}$, then from (3.1.5) and Remark 1.3 (8), we have $d(z,gz) = 0$; i.e., $z = gz$.

Hence, $fz = gz = z$.

The uniqueness of z follows from the inequality (1.11.2). Hence the theorem follows. □

The following is an example in support of Theorem 3.1.

Example 3.2. Let X, E, P, d, φ, f and g be as in Example 1.12.
The pair (f,g) is a Zamfirescu pair with $k = \frac{2}{3}$; and the maps f and g satisfy all the conditions of Theorem 3.1 and 0 is the unique common fixed point of f and g.

The following are corollaries which follow from Theorem 3.1.

Corollary 3.3. *Let (X,d) be a complete cone metric space and P a cone with nonempty interior. Let $f,g : X \to X$ be selfmaps. Suppose that for some constant $k \in (0,1)$ and for every $x,y \in X$,*

$$d(fx,gy) \leq k\ d(x,y).$$

Then f and g have a unique common fixed point in X.

Corollary 3.4. *Let (X,d) be a complete cone metric space and P a cone with nonempty interior. Let $f,g : X \to X$ be selfmaps. Suppose that for some constant $k \in (0,\frac{1}{2})$ and for every $x,y \in X$,*

$$d(fx,gy) \leq k\ [d(x,fx) + d(y,gy)].$$

Then f and g have a unique common fixed point in X.

Corollary 3.5. *Let (X, d) be a complete cone metric space and P a cone with nonempty interior. Let $f, g : X \to X$ be selfmaps. Suppose that for some constant $k \in (0, \frac{1}{2})$ and for every $x, y \in X$,*

$$d(fx, gy) \leq k\ [d(x, gy) + d(y, fx)].$$

Then f and g have a unique common fixed point in X.

Remark 3.6. Corollary 3.3, Corollary 3.4 and Corollary 3.5 are extensions of Theorem 2.3, Theorem 2.4 and Theorem 2.5 of Rezapour and Hamlbarani [4] respectively and hence generalize some results of Huang and Zhang [2] since we do not use the assumption 'normality of cone' in our results.

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