ALBANIAN JOURNAL OF MATHEMATICS
Volume 6, Number 1, Pages 11–19
ISSN: 1930-1235; (**2012**)

ON ISOLATED STRATA OF P-GONAL RIEMANN SURFACES IN THE BRANCH LOCUS OF MODULI SPACES

Gabriel Bartolini
Matematiska institutionen,
Linköpings universitet,
581 83 Linköping, Sweden.

Antonio F. Costa[1]
Dept. Matemáticas Fundamentales,
Facultad de Ciencias, UNED,
28040 Madrid, Spain.

Milagros Izquierdo[2]
Matematiska institutionen,
Linköpings universitet,
581 83 Linköping, Sweden.

Abstract. The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g has orbifold structure, and the set of singular points of such orbifold is the *branch locus* $\mathcal{B}_g$. Given a prime number $p \geq 7$, $\mathcal{B}_g$ contains isolated strata consisting of p-gonal Riemann surfaces for genera $g \geq \frac{3(p-1)}{2}$, that are multiple of $\frac{p-1}{2}$. This is a generalization of the results obtained in [BCI1] for pentagonal Riemann surfaces, and the results of [K] and [CI3] for zero- and one-dimensional isolated strata in the branch locus.

1. Introduction

In this article we study the topology of moduli spaces of Riemann surfaces. The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g, being the quotient of the Teichmüller space by the discontinuous action of the mapping class group, has the structure of a complex orbifold, whose set of singular points is called the

2000 *Mathematics Subject Classification.* 14H99, 20F10, 30F10.
Key words and phrases. Dual Tiling, Connectedness, Rauzy Fractal, Greedy expansion.
[1]Partially supported by MTM2011-23092
[2]Partially supported by the Royal Swedish Academy of Sciences. Work done during a visit to the Institut Mittag-Leffler (Djursholm, Sweden)

branch locus $\mathcal{B}_g$. The branch locus $\mathcal{B}_g$, $g \geq 3$ consists of the Riemann surfaces with symmetry, i. e. Riemann surfaces with non-trivial automorphism group; see [H] and [B]. $\mathcal{B}_g$ admits an (equisymmetric) stratification where each stratum is given by the symmetry of the surfaces in it, i.e. the conjugacy class in the mapping class group of the automorphism group of the surfaces of the stratum ([B]).

Our goal is to study the topology of $\mathcal{B}_g$ through its connectedness, using this equisymmetric stratification. The connectedness of moduli spaces of hyperelliptic, $p-$gonal and real Riemann surfaces has been widely studied, for instance by [BSS], [K] [CI1], [CI2], [CI3], [BCI2], [G], [S], [BCIP] and [BEMS].

Recently Bartolini, Costa and Izquierdo have shown that $\mathcal{B}_g$ is connected only for genera 3, 4, 7, 13, 17, 19 and 59; see [BCI1] and [BCI2]. The authors found isolated strata in $\mathcal{B}_g$ ($g \neq 3, 4, 7, 13, 17, 19, 59$) given by actions of order five and seven. In [BI] it is shown that the strata induced by actions of order two and three belongs to the same connected component of $\mathcal{B}_g$.

A cyclic p-gonal Riemann surface X is a surface that admits a regular covering of degree p on the Riemann sphere. A 2-gonal Riemann surface is called an hyperelliptic Riemann surface.

The main result in this article is that $\mathcal{B}_g$ contains isolated strata consisting of p-gonal Riemann surfaces ($p \geq 7$) of dimension $d \geq 2$ for genus $g = (d+1)(\frac{p-1}{2})$, according to Riemann-Hurwitz's formula.

Given two Riemann surfaces X_1 and X_2, there is a path of quasiconformal deformations taking X_1 to X_2 since $\mathcal{M}_g$ is connected. The result obtained in this article says that if X_1 belongs to one of the isolated strata and X_2 has another type of symmetry, then the path of quasiconformal deformations must contain surfaces without symmetry.

The main result is a generalization of the results obtained in [BCI1] for isolated strata of cyclic pentagonal Riemann surfaces, and of the results in [K], [CI3] for isolated strata of dimension zero and one. As a consequence we give an infinite family of genera for which $\mathcal{B}_g$ has an increasing number of isolated strata.

2. Riemann surfaces and Fuchsian groups

Let X be a Riemann surface and assume that $Aut(X) \neq \{1\}$. Hence $X/Aut(X)$ is an orbifold and there is a Fuchsian group $\Gamma \leq Aut(\mathcal{D})$, such that $\pi_1(X) \lhd \Gamma$ and

$$\mathcal{D} \to X = \mathcal{D}/\pi_1(X) \to X/Aut(X) = \mathcal{D}/\Gamma$$

where $\mathcal{D} = \{z \in \mathbb{C} : \|z\| < 1\}$.

If the Fuchsian group Γ is isomorphic to an abstract group with canonical presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1 \ldots x_k | x_1^{m_1} = \cdots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle, \tag{1}$$

we say that Γ has *signature*

$$s(\Gamma) = (g; m_1, \ldots, m_k). \tag{2}$$

The generators in presentation (1) will be called *canonical generators*.

Let X be a Riemann surface uniformized by a surface Fuchsian group Γ_g, i.e. a group with signature $(g; -)$. A finite group G is a group of automorphisms of X, i.e. there is a holomorphic action a of G on X, if and only if there is a Fuchsian group Δ and an epimorphism $\theta_a : \Delta \to G$ such that $\ker \theta_a = \Gamma_g$. The epimorphism θ_a is the monodromy of the covering $f_a : X \to X/G = \mathcal{D}/\Delta$.

The relationship between the signatures of a Fuchsian group and subgroups is given in the following theorem of Singerman:

Theorem 1. *(Singerman [Si1]) Let Γ be a Fuchsian group with signature (2) and canonical presentation (1). Then Γ contains a subgroup Γ' of index N with signature*

$$s(\Gamma') = (h; m'_{11}, m'_{12}, ..., m'_{1s_1}, ..., m'_{k1}, ..., m'_{ks_k}).$$

if and only if there exists a transitive permutation representation $\theta : \Gamma \to \Sigma_N$ satisfying the following conditions:

1. The permutation $\theta(x_i)$ has precisely s_i cycles of lengths less than m_i, the lengths of these cycles being $m_i/m'_{i1}, ..., m_i/m'_{is_i}$.

2. The Riemann-Hurwitz formula

$$\mu(\Gamma')/\mu(\Gamma) = N.$$

where $\mu(\Gamma)$, $\mu(\Gamma')$ are the hyperbolic areas of the surfaces $\mathcal{D}/\Gamma$, $\mathcal{D}/\Gamma'$.

For $\mathcal{G}$, an abstract group isomorphic to all the Fuchsian groups of signature $s = (h; m_1, ..., m_k)$, the Teichmüller space of Fuchsian groups of signature s is

$$\{\rho : \mathcal{G} \to PSL(2, \mathbb{R}) : s(\rho(\mathcal{G})) = s\}/ \text{ conjugation in } PSL(2, \mathbb{R}) \ = \ T_s.$$

The Teichmüller space T_s is a simply-connected complex manifold of dimension $3g - 3 + k$. The modular group, $M(\Gamma)$, of Γ, acts on $T(\Gamma)$ as $[\rho] \to [\rho \circ \alpha]$ where $\alpha \in M(\Gamma)$. The moduli space of Γ is the quotient space $\mathcal{M}(\Gamma) = T(\Gamma)/M(\Gamma)$, then $\mathcal{M}(\Gamma)$ is a complex orbifold and its singular locus is $\mathcal{B}(\Gamma)$, called the branch locus of $\mathcal{M}(\Gamma)$. If Γ_g is a surface Fuchsian group, we denote $\mathcal{M}_g = T_g/M_g$ and the branch locus by $\mathcal{B}_g$. The branch locus $\mathcal{B}_g$ consists of surfaces with non-trivial symmetries for $g > 2$.

If $X/Aut(X) = \mathcal{D}/\Gamma$ and genus$(X) = g$, then there is a natural inclusion $i : T_s \to T_g : [\rho] \to [\rho']$, where

$$\rho : \mathcal{G} \to PSL(2, \mathbb{R}),\ \pi_1(X) \subset \mathcal{G},\ \rho' = \rho\mid_{\pi_1(X)} : \pi_1(X) \to PSL(2, \mathbb{R}).$$

If we have $\pi_1(X) \lhd \mathcal{G}$, then there is a topological action of a finite group $G = \mathcal{G}/\pi_1(X)$ on surfaces of genus g given by the inclusion $a : \pi_1(X) \to \mathcal{G}$. This inclusion $a : \pi_1(X) \to \mathcal{G}$ produces $i_a(T_s) \subset T_g$.

The image of $i_a(T_s)$ by $T_g \to \mathcal{M}_g$ is $\overline{\mathcal{M}}^{G,a}$, where $\overline{\mathcal{M}}^{G,a}$ is the set of Riemann surfaces with automorphisms group containing a subgroup acting in a topologically equivalent way to the action of G on X given by the inclusion a, see [H], the subset $\mathcal{M}^{G,a} \subset \overline{\mathcal{M}}^{G,a}$ is formed by the surfaces whose full group of automorphisms acts in the topological way given by a. The branch locus, $\mathcal{B}_g$, of the covering $T_g \to \mathcal{M}_g$ can be described as the union $\mathcal{B}_g = \bigcup_{G \neq \{1\}} \overline{\mathcal{M}}^{G,a}$, where $\{\mathcal{M}^{G,a}\}$ is the equisymmetric stratification of the branch locus [B]:

Theorem 2. *(Broughton [B]) Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, G a finite subgroup of the corresponding modular group M_g. Then:*

(1) $\overline{\mathcal{M}}_g^{G,a}$ is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$.

(2) $\mathcal{M}_g^{G,a}$, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in $\overline{\mathcal{M}}_g^{G,a}$.

There are finitely many strata $\mathcal{M}_g^{G,a}$.

An isolated stratum $\mathcal{M}^{G,a}$ in the above stratification is a stratum that satisfies $\overline{\mathcal{M}}^{G,a} \cap \overline{\mathcal{M}}^{H,b} = \varnothing$, for every group H and action b on surfaces of genus g. Thus $\overline{\mathcal{M}}^{G,a} = \mathcal{M}^{G,a}$

Since each non-trivial group G contains subgroups of prime order, we have the following remark:

Remark 3. *(Cornalba [C])*

$$\mathcal{B}_g = \bigcup_{p\ prime} \overline{\mathcal{M}}^{C_p,a}$$

where $\overline{\mathcal{M}}^{C_p,a}$ is the set of Riemann surfaces of genus g with an automorphism group containing C_p, the cyclic group of order p, acting on surfaces of genus g in the topological way given by a.

3. Isolated strata of p-gonal Riemann surfaces

Definition 4. *A Riemann surface X is said to be p-gonal if it admits a p-sheeted covering $f : X \to \widehat{\mathbb{C}}$ onto the Riemann sphere. If f is a cyclic regular covering then X is called cyclic p-gonal. The covering f will be called the (cyclic) p-gonal morphism.*

A cyclic p-gonal Riemann surface admits an equation of the form $y^p = P(x)$.
By Lemma 2.1 in [A], if the surface X_g has genus $g \geq (p-1)^2+1$, then the p-gonal morphism is unique.

We can characterize cyclic p-gonal Riemann surfaces using Fuchsian groups. Let X_g be a Riemann surface, X_g admits a cyclic p-gonal morphism f if and only if there is a Fuchsian group Δ with signature $(0;\overbrace{p,...,p}^{\frac{2g}{p-1}+2})$ and an index p normal surface subgroup Γ of Δ, such that Γ uniformizes X_g; see [CI4], [CI].

We have the following algorithm to recognize cyclic p-gonal surfaces: A surface X_g admits a cyclic p-gonal morphism f if and only if there is a Fuchsian group Δ with signature $(0; m_1, ..., m_r)$, an order p automorphism $\alpha : X_g \to X_g$, such that $\langle\alpha\rangle \leq G = Aut(X_g)$, and an epimorphism $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$ in such a way that $\theta^{-1}(\langle\alpha\rangle)$ is a Fuchsian group with signature $(0;\overbrace{p,...,p}^{\frac{2g}{p-1}+2})$. Furthermore the p-gonal morphism f is unique if and only if $\langle\alpha\rangle$ is normal in G (see [G]), and Wootton [W] has proved the following:

Lemma 5. *(Wootton [W]) With the notation above. If $G > C_p$, then $N_G(C_p) > C_p$.*

Isolated strata $\overline{\mathcal{M}}^{C_p,a} = \mathcal{M}^{C_p,a}$ of cyclic p-gonal surfaces correspond to maximal actions of the cyclic group C_p. Isolated strata of dimension 0 where given [K], isolated strata of dimension 1 were studied in [CI3]. We find here isolated strata of any dimension, consisting of p-gonal surfaces also.

Theorem 6. *Let p be a prime number at least seven and let $d \geq 2$. Then there are isolated strata of dimension d consisting of p-gonal surfaces in $\mathcal{B}_g$ if and only if $g = (d+1)(\frac{p-1}{2})$.*

Proof. First of all, an equisymmetric stratum $\mathcal{M}^{C_p,a}$ in $\mathcal{B}_p$ of dimension $d \geq 2$ of p-gonal Riemann surfaces is given by a monodromy $\theta : \Delta(0;\overbrace{p,\ldots,p}^{d+3}) \to C_p$, with Δ a Fuchsian group with maximal signature; see [Si2]. Then, a generic surface X in $\mathcal{M}^{C_p,a}$ will have $C_p = Aut(X)$. The dimension of the stratum is $d = \frac{2g-p+1}{p-1}$ by the Riemann-Hurwitz formula Thus $g = (d+1)(\frac{p-1}{2})$.
If a surface in the stratum has larger automorphism group G, then, by Lemma 5, we can assume that C_p is normal in G by considering $C_p < N_G(C_p)$.
Let X_g, be a p-gonal surface, such that $X_g \in \overline{\mathcal{M}}_g^{C_p,a}$ for some action a, let $\langle\alpha\rangle$ be the group of p-gonal automorphisms of X_g. Consider an automorphism $b \in Aut(X) \setminus \langle\alpha\rangle$, by Lemma 5 and [G], b induces an automorphism $\hat{b}$ of order $t \geq 2$ on the Riemann sphere $X_g/\langle a\rangle = \widehat{\mathbb{C}}$ according to the following diagram

$$\begin{array}{ccc} X_g = \mathcal{D}/\Gamma_g & \stackrel{b}{\to} & X_g = \mathcal{D}/\Gamma_g \\ f_a \downarrow & & \downarrow f_a \\ X_g/\langle\alpha\rangle = \widehat{\mathbb{C}}(P_1,\ldots,P_k) & \stackrel{\hat{b}}{\to} & X_g/\langle\alpha\rangle = \widehat{\mathbb{C}}(P_1,\ldots,P_k), \end{array}$$

where Γ_g is a surface Fuchsian group and $f_a : X_g = \mathcal{D}/\Gamma_g \to X_g/\langle\alpha\rangle$ is the p-gonal morphism induced by the group of automorphisms $\langle\alpha\rangle$ with action a. $S = \{P_1,\ldots,P_k\}$ is the branch set in $\widehat{\mathbb{C}}$ of the morphism f_a with monodromy $\theta_a : \Delta(0;p,\stackrel{d+3}{\ldots},p) \to C_p$ defined by $\theta_a(x_i) = \alpha^{r_i}$, where $r_i \in \{1,\ldots,p-1\}$ for $1 \leq i \leq d+3$.

Now, $\hat{b}$ induces a permutation on S that either takes singular points with monodromy α^j to points with monodromy $\alpha^{\beta(j)}$, with β an automorphism of C_p, or it acts on each subset formed by points in S with same monodromy α^{r_j}.

We construct monodromies $\theta : \Delta(0;p,\stackrel{d+3}{\ldots},p) \to C_p = \langle\alpha\rangle$, where $d = \frac{2g}{p-1} - 2 \geq 2$ by the Riemann-Hurwitz formula. We separate the monodromies in cases according to the congruence of d modulus p.

(1) $d \equiv r \not\equiv 0, 2, p-2, p-1 \, mod(p)$
  $\theta : \Delta(0;p,\stackrel{d+3}{\ldots},p) \to C_p$ is defined by
  $\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^2$, $\theta(x_{d+2}) = \alpha^{p-2}$, $\theta(x_{d+3}) = \alpha^{p-r}$.
(2) $d \equiv 0 \, mod(p)$
  $\theta : \Delta(0;p,\stackrel{d+3}{\ldots},p) \to C_p$ is defined by
  $\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^3$, $\theta(x_{d+2}) = \alpha^5$, $\theta(x_{d+3}) = \alpha^{p-8}$.
(3) $d \equiv 2 \, mod(p)$
  $\theta : \Delta(0;p,\stackrel{d+3}{\ldots},p) \to C_p$ is defined by
  $\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^3$, $\theta(x_{d+2}) = \alpha^{p-3}$, $\theta(x_{d+3}) = \alpha^{p-2}$.
(4) $d \equiv p-2 \, mod(p)$
  $\theta : \Delta(0;p,\stackrel{d+3}{\ldots},p) \to C_p$ is defined by
  $\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^3$, $\theta(x_{d+2}) = \alpha^{p-3}$, $\theta(x_{d+3}) = \alpha^2$.
(5) $d \equiv p-1 \, mod(p)$
  $\theta : \Delta(0;p,\stackrel{d+3}{\ldots},p) \to C_p$ is defined by
  $\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^4$, $\theta(x_{d+2}) = \alpha^5$, $\theta(x_{d+3}) = \alpha^{p-8}$.

(Notice that $p-8=6$ when $p=7$ in cases 2 and 5)
We see that the given epimorphisms force $\hat{b}$ to be the identity on $\widehat{\mathbb{C}}$. Thus, the surfaces X_g do not admit a larger group of automorphisms than $C_p = \langle\alpha\rangle$ and the equisymmetric strata given by the monodromies above are isolated.

□

Theorem 6 generalizes de results obtained in [BCI1] for isolated strata of pentagonal Riemann surfaces, the results in [CI3] for one-dimensional isolated strata, and the results in [K] for isolated Riemann surfaces. Kulkarni [K] showed that a branch locus $\mathcal{B}_g$ contains isolated Riemann surfaces if and only if $g = 2$ or $g = \frac{p-1}{2}$, with $p \geq 11$ a prime number. The isolated Riemann surfaces are cyclic p-gonal surfaces. Costa and Izquierdo [CI3] showed that $\mathcal{B}_g$ contains one-dimensional isolated strata if and only if $g = p - 1$, with $p \geq 11$ a prime number.

Remark 7. *The isolated strata of heptagonal surfaces with dimension $\frac{g}{3} - 1$ in $\mathcal{B}_g$ obtained here are different of the isolated strata of heptagonal surfaces and dimension $\frac{g}{3} - 1$ obtained in* [BCI2] *since the actions determined by the monodromies are not topologically equivalent, see* [H].

In [BCI1] we showed that $\mathcal{B}_g$ contains isolated strata of cyclic pentagonal surfaces for all even genera greater or equal eighteen. In [BI] (see also [Bo] and [BCIP]) it is shown that the $\mathcal{B}_2$ contains one isolated pentagonal Riemann surface and that $\mathcal{B}_4$, $\mathcal{B}_6$ and $\mathcal{B}_8$ do not contain isolated strata of pentagonal Riemann surfaces. We study the remaining branch loci in the following proposition:

Proposition 8.

(1) $\mathcal{B}_{10}$, $\mathcal{B}_{14}$ and $\mathcal{B}_{16}$ contain isolated strata of cyclic pentagonal Riemann surfaces.
(2) $\mathcal{B}_{12}$ does not contain isolated strata of cyclic pentagonal Riemann surfaces.

Proof. (1) Consider monodromies:

$\theta_1 : \Delta(0; 5, \overset{7}{\ldots}, 5) \to C_5 = \langle\alpha\rangle$ defined by $\theta_1(x_1) = \theta_1(x_2) = \theta_1(x_3) = \alpha$, $\theta_1(x_4) = \alpha^2$, $\theta_1(x_5) = \theta_1(x_6) = \alpha^3$, $\theta_1(x_7) = \alpha^4$,

$\theta_2 : \Delta(0; 5, \overset{9}{\ldots}, 5) \to C_5 = \langle\alpha\rangle$ defined by $\theta_2(x_i) = \alpha, 1 \leq i \leq 6$, $\theta_2(x_7) = \alpha^2$, $\theta_2(x_8) = \alpha^3$, $\theta_2(x_9) = \alpha^4$,

$\theta_3 : \Delta(0; 5, \overset{10}{\ldots}, 5) \to C_5 = \langle\alpha\rangle$ defined by $\theta_3(x_1) = \alpha$, $\theta_3(x_2) = \alpha^2$, $\theta_3(x_3) = \cdots = \theta_3(x_5)\alpha^3$, $\theta_3(x_6) = \cdots = \theta_3(x_{10}) = \alpha^4$

With the same argument as in Theorem 6 we see that θ_1, θ_2 and θ_3 induce isolated strata in $\mathcal{B}_{10}$, $\mathcal{B}_{14}$ and $\mathcal{B}_{16}$ respectively.

(2) Case $\mathcal{B}_{12}$. The only possible monodromies $\theta : \Delta(0, 5, \overset{8}{\ldots}, 5) \to C_5 = \langle\alpha\rangle$ are, up to an automorphism of C_5 and permuting the order of the generators of Δ:

i) $\theta(x_1) = \cdots = \theta(x_5) = \alpha$, $\theta(x_6) = \alpha^2$, $\theta(x_7) = \theta(x_8) = \alpha^4$;
ii) $\theta(x_1) = \cdots = \theta(x_5) = \alpha$, $\theta(x_6) = \alpha^4$, $\theta(x_7) = \theta(x_8) = \alpha^3$;
iii) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_6) = \cdots = \theta(x_8) = \alpha^4$;
iv) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_5) = \alpha^2$, $\theta(x_6) = \theta(x_7) = \theta(x_8) = \alpha^4$;
v) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_5) = \alpha^2$, $\theta(x_6) = \theta(x_7) = \theta(x_8) = \alpha^3$;
vi) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_5) = \theta(x_6) = \alpha^2$, $\theta(x_7) = \alpha^3$, $\theta(x_8) = \alpha^4$;
vii) $\theta(x_1) = \theta(x_2) = \theta(x_3) = \alpha$, $\theta(x_4) = \theta(x_5) = \theta(x_6) = \alpha^2$, $\theta(x_7) = \theta(x_8) = \alpha^3$;
viii) $\theta(x_1) = \theta(x_2) = \theta(x_3) = \alpha$, $\theta(x_4) = \alpha^2$, $\theta(x_5) = \alpha^3$, $\theta(x_6) = \theta(x_7) = \theta(x_8) = \alpha^4$;

ix) $\theta(x_1) = \theta(x_2) = \alpha$, $\theta(x_3) = \theta(x_4) = \alpha^2$, $\theta(x_5) = \theta(x_6) = \alpha^3$, $\theta(x_7) = \theta(x_8) = \alpha^4$.

With the argument in the proof of Theorem 6 the action of C_5 on the pentagonal surfaces $\mathcal{D}/Ker(\theta)$ can be extended to the action of a larger group. For instance the action of C_5 in case **ix)** can be extended to an action of C_{10}, D_5 or $C_5 \rtimes C_4$.

□

Remark 9. *Theorem 6 and Porposition 8 can be interpreted geometrically as follows: Let (X_g^1, C_p) and (X_g^2, G) be two Riemann surfaces with symmetry, where X_1 belongs to one of the isolated strata of cyclic p-gonal surfaces in $\mathcal{B}_g$ and X_g^2 has another symmetry. Then any path of quasiconformal deformations joining X_g^1 and X_g^2 must contain surfaces without symmetry.*

We consider the existence of several isolated equisymmetric strata in branch loci. Let $5 \leq p_1 < p_2 < \cdots < p_r$ be prime numbers. We define $\lambda = l.c.m.(\frac{p_i-1}{2})_{i=1}^r$. As a consequence of Theorem 6 , Theorem 3.6 in [K] and Theorem 5 in [CI3] we obtain:

Theorem 10. *Let $5 \leq p_1 < p_2 < \cdots < p_r$ be prime numbers. Then, for all $g = k\,\lambda$, $k \geq 1$ and $g > 12$, the branch locus $\mathcal{B}_g$ contains r isolated strata formed by cyclic p_i-gonal Riemann surfaces, $1 \leq i \leq r$.*

Proof. Observe that the conditions of Theorem 6 are satisfied if $g \geq \frac{3}{2}(p_r - 1)$. The conditions of Theorem 5 in [CI3] and Theorem 6 are satisfied if $g = p_r - 1$. Finally the conditions of Theorem 6, Theorem 5 in [CI3] and Theorem 3.6 in [K] are satisfied if $g = \frac{p_r-1}{2}$. The dimension of the isolated strata of cyclic p_i-gonal surfaces is $d_i = \frac{2g}{p_i-1} - 1$ by the Riemann-Hurwitz formula.

$\mathcal{B}_{12}$ does not contain isolated strata of cyclic pentagonal Riemann surfaces, it contains isolated strata of cyclic heptagonal Riemann surfaces.

□

As a consequence we have:

Corollary 11. *Given a number $r \in \mathbb{N}$, there is an infinite number of genera g such that $\mathcal{B}_g$ contains at least r isolated equisymmetric strata.*

We finish with some examples for small genera.

3.1. **Examples.**

(1) By Theorem 5 in [CI3] and Proposition 8, $\mathcal{B}_{10}$ contains one isolated stratum of cyclic pentagonal surfaces of dimension four, and one 1-dimensional startum of cyclic 11-gonal surfaces.
(2) By Theorem 6 and Theorem 5 in [CI3], the smallest genus for which the branch locus contains isolated strata of cyclic heptagonal and 13-gonal Riemann surfaces is twelve.The dimensions of the isolated strata are 3 and 1 respectively.
(3) By Theorem 10, $\mathcal{B}_{20}$ contains both isolated strata of cyclic pentagonal and 11-gonal Riemann surfaces. The dimensions of the isolated strata are 9 and 3 respectively. By [K], $\mathcal{B}_{20}$ contains isolated Riemann surfaces that are cyclic 41-gonal.

(4) The smallest genus for which the branch locus contains both isolated strata of cyclic heptagonal and 11-gonal Riemann surfaces is fifteen. The dimensions of the isolated strata are 3 and 2 respectively. By [K], $\mathcal{B}_{15}$ contains isolated Riemann surfaces that are cyclic 31-gonal.

(5) The smallest genus g for which the branch locus $\mathcal{B}_g$ contains both isolated strata of cyclic pentagonal and heptagonal Riemann surfaces is eighteen. The dimensions of the strata are 8 and 5 respectively. It contains also isolated strata of cyclic 13-gonal Riemann surfaces of dimension 3. By [CI3] and [K], $\mathcal{B}_{18}$ contains one-dimensional isolated strata of cyclic 19-gonal surfaces and isolated cyclic 37-gonal Riemann surfaces.

(6) By Theorem 10, $\mathcal{B}_{24}$ contains isolated strata of cyclic pentagonal, heptagonal, 13-gonal and 17-gonal Riemann surfaces. The dimensions of the isolated strata are 11, 7, 3 and 2 respectively.

(7) The smallest genus for which the branch locus contains isolated strata of cyclic pentagonal, heptagonal and 11-gonal Riemann surfaces is thirty, the dimensions of these strata are 14, 9 and 5 respectively. By Theorem 10, $\mathcal{B}_{30}$ contains also isolated strata of cyclic 13-gonal Riemann, surfaces with dimension 4. By [CI3] and [K], $\mathcal{B}_{30}$ contains one-dimensional isolated strata of cyclic 31-gonal surfaces and isolated cyclic 61-gonal Riemann surfaces.

(8) $\mathcal{B}_{60}$ contains isolated strata of cyclic pentagonal, heptagonal, 11-gonal, 13-gonal, 31-gonal, 41-gonal, 61-gonal surfaces, with dimensions 29, 19, 11, 9, 3, 2 and 1 respectively.

(9) $\mathcal{B}_{1000}$ contains isolated strata of cyclic pentagonal, 11-gonal, 17-gonal, 41-gonal, 101-gonal, 251-gonal and 401-gonal surfaces, with dimensions 499, 199, 124, 49, 19, 7 and 4 respectively.

(10) $\mathcal{B}_{2012}$ contains 1005-dimensional isolated strata of cyclic pentagonal surfaces.

References

[A] Accola, R. D. M. (1984) On cyclic trigonal Riemann surfaces. I. Trans. Amer. Math. Soc. 283 no. 2, 423–449.

[BCI1] Bartolini, G., Costa, A.F., Izquierdo, M., (2012) On isolated strata of pentagonal Riemann surfaces in the branch locus of moduli spaces, Contemp. Maths. 572, 19-24.

[BCI2] Bartolini, G., Costa, A.F., Izquierdo, M., (2012) On the connectivity of the branch locus of moduli spaces of Riemann surfaces, Preprint.

[BCIP] Bartolini, G., Costa, A.F., Izquierdo, M., Porto, A.M., (2010) On the connectedness of the branch locus of the moduli space of Riemann surfaces, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 104 no.1, 81-86.

[BI] Bartolini, G., Izquierdo, M. (2012) On the connectedness of branch loci of moduli spaces of Riemann surfaces of low genus. Proc. Amer. Math. Soc. 40, 35-45.

[Bo] Bolza, O. (1888) On binary sextics with linear transformations between themselves, Amer. J. Math. 10, 47–70.

[B] Broughton, S. A. (1990) The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups. Topology Appl. 37, 101–113.

[BEMS] Bujalance, E., Etayo, J. J., Martnez, E., Szepietowski, B. (2011) On the connectedness of the branch loci of nonorientable unbordered Klein surfaces of low genus. Preprint

[BSS] Buser, P., Seppälä, M. , Silhol, R.(1995) Triangulations and moduli spaces of Riemann surfaces with group actions. Manuscripta Math. 88, 209-224.

[CI1] Costa, A. F., Izquierdo, M. (2002) On the connectedness of the locus of real Riemann surfaces. Ann. Acad. Sci. Fenn. Math. 27, 341-356.

[CI2] Costa, A. F., Izquierdo, M. (2010) On the connectedness of the branch locus of the moduli space of Riemann surfaces of genus 4. Glasg. Math. J. 52, no. 2, 401-408.

[CI3] Costa, A. F., Izquierdo, M. (2012) On the existence of connected components of dimension one in the branch loci of moduli spaces of Riemann surfaces. Math Scand. 111, 1-12.

[CI4] Costa, A. F., Izquierdo, M. (2004) Symmetries of real cyclic p-gonal Riemann surfaces. Pacific J. Math. **213** 231–243.

[CI] Costa, A. F., Izquierdo, M. (2006) On real trigonal Riemann surfaces. Math. Scand. **98** (2006) 53-468.

[C] Cornalba, M. (1987) On the locus of curves with automorphisms. Annali di Matematica Pura e Applicata (4) 149, 135-151.

[G] González-Díez, G. (1995). On prime Galois covering of the Riemann sphere. Ann. Mat. Pure Appl. 168, 1-15.

[H] Harvey, W. (1971) On branch loci in Teichmüller space. Trans. Amer. Math. Soc. 153, 387-399.

[K] Kulkarni, R. S. (1991) Isolated points in the branch locus of the moduli space of compact Riemann surfaces. Ann. Acad. Sci. Fen. Ser. A I MAth. 16, 71-81.

[S] Seppälä, M. (1990) Real algebraic curves in the moduli space of complex curves. Comp. Math., 74, 259-283.

[Si1] Singerman, D. (1970) Subgroups of Fuchsian groups and finite permutation groups Bull. London Math. Soc. 2, 319-323.

[Si2] Singerman, D. (1972) Finitely maximal Fuchsian groups. J. London Math. Soc. 6, 29-38.

[W] Wootton, A. (2007) The full automorphism group of a cyclic p-gonal surface. J. Algebra 312, 29-38.